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How far away is the harmonic mean from the homogenized matrix?

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We present a Hilbert space perspective to homogenization of standard linear evolutionary boundary value problems in mathematical physics and give a unified treatment for (non-)periodic homogenization problems in thermodynamics, elasticity, electro-magnetism and coupled systems thereof. The approach permits the consideration of memory problems. We show that the limiting equation is well-posed and causal and prove that the principal physical phenomenon remains unchanged after the homogenization process.

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0 Introduction

When studying evolutionary equations of mathematical physics, we are frequently confronted with the issue of heterogeneous materials that result in variable coefficients in the (partial differential) equations. If the heterogeneities occur only on a small scale, these coefficients might be highly oscillatory resulting in the need for significant computational effort to solve these equations. That is why one is interested in the behavior of the solutions when the (small) scale tends to zero. Moreover, if the solutions converge in some appropriate sense, it is of interest to know whether the limit is a solution of a similar equation and to determine the coefficients. The main objective in homogenization theory is to show the convergence of the solutions as the small scale tends to zero and to derive the limit equation. If one assumes periodicity in the coefficients, many results are available for particular equations, see e.g. [4, 8, 26] as general references. In the non-periodic case one cannot expect a similar behavior as very simple equations show, see e.g. [29, p. x, equation (*)]. In this note, we discuss a general compactness result, proving that the existence of a limit equation (at least for subsequences) is independent of any periodic behavior of the coefficients. Further, we show that this compactness result may be easily applied to coupled systems that arise in the area of so-called multi-physics and to equations with memory. Apart from that our formulation of the problem is somewhat different from the classical approaches to homogenization theory. Starting out with a formulation of an equation of mathematical physics being first order in time *and* space, we deduce a different representation of the limiting equations, see also Section 1 for a detailed discussion. One may prefer this strategy if there is no appropriate second order formulation.

In the literature, there are many techniques available that allow the study of homogenization in the non-periodic case. We mention here the method of H -convergence in the sense of [11, 15, 26] or [8, Definition 13.3], which is well-suited for elliptic equations. The method of Γ -convergence is tailored for variational integrals and optimization problems related to them, see e.g. [5, 6, 14]. G -convergence and two-scale-convergence, see respectively [9, 24, 36] and [2, 16, 34], are more general concepts since they can be applied to many equations of mathematical physics. Whereas the notion of G -convergence in the very general sense of [36, p.74] (see also Section 4 in this paper) may lead to results that are not precise enough, the notion of two-scale convergence may be too restrictive to give useful results for the general situation. In particular, n -scale convergence needs additional consideration, cf. [12, p.2] or [1]. For a possible way of dealing with specific non-periodic coefficients, we refer to the generalization of two-scale convergence in [17, 18]. We give a structural way of discussing homogenization problems, which was introduced in [29, 30] with extensions in [31]. The idea is to use the observation of [21, 22] to formulate homogenization problems in an operator-theoretic language.

The core of [21] and [22, Chapter 6] is that many linear evolutionary problems in mathematical physics may be written as

$$\partial_0 w + Au = f, \tag{0.1}$$

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completed by some initial conditions and a *constitutive relation* or *material law* \mathcal{M} , being a continuous linear operator linking w and u via the equation

$$\mathcal{M}u = w. \quad (0.2)$$

Here ∂_0 is a realization of the time-derivative as a normal *continuously invertible* operator, A is a skew-selfadjoint operator in some Hilbert space H modelling the spatial derivatives and f is a given right-hand side. We assume here that \mathcal{M} may be represented as a function of ∂_0^{-1} . This leads to time-translation-invariant and causal material laws. We may substitute the constitutive relation (0.2) into (0.1) to give

$$\partial_0 \mathcal{M}u + Au = f. \quad (0.3)$$

This way of considering partial differential equations may be unusual. However, this point of view gives a convenient way to discuss many problems in homogenization theory in a unified manner. Indeed, the material coefficients \mathcal{M} consist of the conductivity matrix if one discusses the heat equation or the electric permittivity, the conductivity and the magnetic permeability in case of Maxwell's equations. Thus, the discussion of homogenization problems in this setting boils down to the discussion of stability under perturbations of \mathcal{M} in a suitable topology in equation (0.3).

Our model of homogenization theory may now be stated as follows, see also Section 1 for a more detailed discussion. Consider a sequence of material laws $(\mathcal{M}_n)_n$ and corresponding solutions $(u_n)_n$ of the equation

$$\partial_0 \mathcal{M}_n u_n + Au_n = f.$$

We ask, whether the sequence $(u_n)_n$ converges to some limit v and whether there is a material law \mathcal{N} , such that v solves

$$\partial_0 \mathcal{N}v + Av = f.$$

In [31], this question was satisfactorily answered in the case of A having compact resolvent, which is the case if considering the heat equation or the wave equation on a bounded open set Ω satisfying some standard geometric requirements. In addition, it is assumed that the material law has a special structure such that the respective equation may also be written as a particular second order system. With this approach it is possible to consider fractional time-derivatives or ordinary differential equations as constitutive relations, see [31, Theorem 4.3 and Theorem 4.5]. In this paper, we generalize these results in the sense that we only require A to have compact resolvent when restricted to a domain orthogonal to the nullspace of A , i.e., instead of assuming that¹ $(D(A), |\cdot|_A) \hookrightarrow (H, |\cdot|_H)$, we only assume the *nullspace-compactness-property* (or *(NC)-property* for short) see Corollary 4.7

¹For Hilbert spaces H_1, H_2 and a linear operator $A : D(A) \subseteq H_1 \rightarrow H_2$ with domain $D(A)$, we denote the norm in the Hilbert space H_1 by $|\cdot|_{H_1}$ and the graph norm of A by $|\cdot|_A$. If H_1 is continuously embedded in H_2 , we write $H_1 \hookrightarrow H_2$ or $(H_1, |\cdot|_{H_1}) \hookrightarrow (H_2, |\cdot|_{H_2})$. If this embedding is compact we

below for the definition. A relevant example for an operator satisfying the (NC) -property is the Maxwell operator $A = \begin{pmatrix} 0 & \text{curl} \\ -\text{curl}_0 & 0 \end{pmatrix}$ in $L_2(\Omega)^6$, where curl_0 is the curl operator with electric boundary condition and $\Omega \subseteq \mathbb{R}^3$ is a bounded domain satisfying additional geometric requirements, cf. [35, Theorem 2] or [19, 23]. Furthermore, in [29] it is shown that the operator $\begin{pmatrix} 0 & \text{div} \\ \text{grad}_0 & 0 \end{pmatrix}$ defined in $L_2(\Omega)^{N+1}$ satisfies the (NC) -property, where $\Omega \subseteq \mathbb{R}^N$ is open and bounded, grad_0 is the distributional gradient in $L_2(\Omega)$ with domain equal to $W_{2,0}^1(\Omega)$. The index “0” refers to Dirichlet boundary conditions and div is the negative adjoint of grad_0 . The same reasoning can be applied to the spatial operator of the elastic equations $\begin{pmatrix} 0 & \text{Grad}^* \\ -\text{Grad} & 0 \end{pmatrix}$, where Grad is the symmetrized gradient as defined in [31, Definition 4.9] with some boundary conditions on a bounded domain Ω satisfying suitable geometric requirements, cf. e.g. [33, Theorem 2]. Thus, our main theorem, Theorem 4.5, may be seen as a general theorem giving a compactness result for the homogenization of (coupled) equations in mathematical physics. We shall also mention that the results obtained in this article not only generalize [29, Theorem 2.3.14] but improve the representation of the homogenized equations. Moreover, we show that the homogenized equations satisfy the assumptions of Theorem 2.1, i.e., we have a solution theory for these equations and causality of the solution operator. A detailed discussion of our main result is given in Section 1. In this section we also give an account of the ideas used and compare it to other strategies in the literature.

We sketch our plan to achieve our main result as follows. In Section 2, we discuss the mathematical framework of evolutionary equations and recall the main theorem of [21]. Section 3 sketches the ideas, definitions and main theorems of [30] and [31]. In Section 4 we present our main result, Theorem 4.5. We show optimality of this theorem by means of counterexamples. The proofs in Section 4 also refer to results from Section 6, where some technical lemmas are provided. The results from Section 6 are needed to prove well-posedness of the limiting equation constructed in Theorem 4.5. Before, however, completing the proof of Theorem 4.5 in Section 6, we apply Theorem 4.5 to some examples from mathematical physics in order to illustrate our findings in Section 5.

We indicate weak convergence in a Hilbert space by ‘ \rightharpoonup ’ or ‘w-lim’. Norm-convergence will be denoted by ‘ \rightarrow ’ if not specified differently. The underlying scalar field of any vector space discussed here is \mathbb{C} .

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Starting out with a different formulation of the problem similar to (0.3), we derive homogenization results for a large class of partial differential equations. The homogenized

write $H_1 \hookrightarrow H_2$ or $(H_1, |\cdot|_{H_1}) \hookrightarrow (H_2, |\cdot|_{H_2})$.

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coefficients, however, have a different representation as in classical results in the literature, cf. e.g. [4, 24]. We illustrate the difference between the classical approach and the approach considered here with the heat equation (cf. e.g. [22, p. 350]):

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Denote by $\theta: (0, \infty) \times \Omega \rightarrow \mathbb{R}$ the temperature distribution and by $q: (0, \infty) \times \Omega \rightarrow \mathbb{R}^N$ the heat flux. The heat equation is a system of the two equations

$$\begin{cases} \partial_0 \theta + \operatorname{div} q = f \\ q = -\kappa \operatorname{grad} \theta. \end{cases} \quad (1.1)$$

Here $\partial_0 \theta$ denotes the time-derivative of θ , f is a given source term and $\kappa: \Omega \rightarrow \mathbb{R}^{N \times N}$ is a bounded function being the (material dependent, symmetric) conductivity tensor satisfying $\kappa(x) \geq c$ for some $c > 0$ and all $x \in \Omega$. The first equation in (1.1) is called the *heat flux balance* and the second one *Fourier's law*. The system is completed by boundary and initial conditions. For simplicity, let us assume Dirichlet boundary conditions for θ and zero initial conditions. Similar to the introduction, as a reminder for Dirichlet boundary conditions, we shall write grad_0 instead of grad . Of course, θ and q are the unknowns in the system.

The classical way of discussing the heat equation is to plug Fourier's law into the heat flux balance in order to arrive at

$$\partial_0 \theta - \operatorname{div} \kappa \operatorname{grad}_0 \theta = f. \quad (1.2)$$

Now assume that κ has an extension to \mathbb{R}^N such that $\kappa(x + e_j) = \kappa(x)$ for all $x \in \mathbb{R}^N$, $e_j := (\delta_{ij})_{i \in \{1, \dots, N\}}$, $j \in \{1, \dots, N\}$. In other words, assume that κ is $(0, 1)^N$ -periodic. In homogenization theory one is now interested in the behavior of solutions coming from equations having highly oscillating coefficients. A possible way to model that is to discuss a sequence of problems, i.e., for $n \in \mathbb{N}$ consider the solutions (θ_n, q_n) and θ_n of the following respective equations

$$\begin{cases} \partial_0 \theta_n + \operatorname{div} q_n = f \\ q_n = -\kappa(n \cdot) \operatorname{grad}_0 \theta_n. \end{cases} \quad (1.3)$$

and

$$\partial_0 \theta_n - \operatorname{div} \kappa(n \cdot) \operatorname{grad}_0 \theta_n = f. \quad (1.4)$$

Standard a priori estimates imply that (possibly after passing to a subsequence) $(\theta_n, q_n)_n$ and $(\theta_n)_n$ converge weakly to some functions (θ, q) and θ , respectively. In order to determine θ , classical results derive the temperature distribution to be the solution of the equation $\partial_0 \theta - \operatorname{div} \kappa_0 \operatorname{grad}_0 \theta = f$ with the homogenized (constant-coefficient-)matrix κ_0 from equation (1.4). A main difficulty to overcome in that approach is the question of how to perform the limit in $\int_{\Omega} \kappa(nx) \operatorname{grad}_0 \theta_n(t, x) \cdot \operatorname{grad}_0 \theta_n(t, x) \, dx$. A possible way to deduce convergence of the integral expression to $\int_{\Omega} \kappa_0 \operatorname{grad}_0 \theta(t, x) \cdot \operatorname{grad}_0 \theta(t, x) \, dx$ is the famous “div-curl-lemma” or the “compensated compactness”, which is due to Murat and Tartar, see e.g. [15, 25], or [26, 7]. The strategy of doing so is well-established and can also be applied to other cases such as linearized elasticity.

Starting out with the sequence of equations given by (1.3), we propose another way of deducing the limiting equation. In the following lines we only sketch the ideas as they are rigorously proven in the forthcoming sections. Written in a block-operator-matrix-form the equations (1.3) read as

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \kappa(n\cdot)^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \text{div} \\ \text{grad}_0 & 0 \end{pmatrix} \right) \begin{pmatrix} \theta_n \\ q_n \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \quad (1.5)$$

Following the introduction and recalling that ∂_0 can be realized as a continuously invertible operator, the latter equations are clearly of the form (0.3) with $\mathcal{M}_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \partial_0^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \kappa(n\cdot)^{-1} \end{pmatrix}$ and $A = \begin{pmatrix} 0 & \text{div} \\ \text{grad}_0 & 0 \end{pmatrix}$. Note that, due to the Dirichlet boundary conditions, A is a skew-selfadjoint operator in $L_2(\Omega)^{N+1}$ since $\text{div}^* = -\text{grad}_0$. Having a homogenization result for A having compact resolvent at hand (see [29, Theorem 3.5] or Theorem 4.3 in this article), we want to deduce our homogenization result from the compact resolvent case as treated in Theorem 4.3. Due to the infinite-dimensional nullspace of div in spatial dimensions greater than 1, the operator $\begin{pmatrix} 0 & \text{div} \\ \text{grad}_0 & 0 \end{pmatrix}$ never has a compact resolvent if $N \geq 2$. But the domain of grad_0 equals $W_{2,0}^1(\Omega)$, which is, if endowed with the graph norm, compactly embedded into $L_2(\Omega)$ due to the selection theorem by Rellich and Kondrachov (keep in mind that Ω is assumed to be bounded). In that sense the operator grad_0 is the ‘good’ part, whereas div is the ‘bad’ one. To overcome this problem the idea is to restrict A to a domain being orthogonal to the nullspace of A . Due to Dirichlet boundary conditions the operator grad_0 is one-to-one, thus $N(A)$, the nullspace of A , equals $\{0\} \oplus N(\text{div}) \subseteq L_2(\Omega) \oplus L_2(\Omega)^N$. Since $\text{div}^* = -\text{grad}_0$, we have $L_2(\Omega)^N = \overline{R(\text{grad}_0)} \oplus N(\text{div})$. Using Poincaré’s inequality, we deduce that the range of grad_0 is closed in $L_2(\Omega)^N$. Hence,

$$L_2(\Omega)^N = R(\text{grad}_0) \oplus N(\text{div}).$$

Along that decomposition of $L_2(\Omega)^N$, we decompose the heat flux q_n from equation (1.5) as $q_n = q_n^1 + q_n^2$ with $q_n^1 \in R(\text{grad}_0)$ and $q_n^2 \in N(\text{div})$. A convenient way of writing the operator A acting on the three components (θ_n, q_n^1, q_n^2) is by introducing the operator $P: L_2(\Omega)^N \rightarrow R(\text{grad}_0)$, which maps any $g \in L_2(\Omega)^N$ to its orthogonal projection Pg in the range of grad_0 . Looking at P in that way, the adjoint of P is the identity with domain $R(\text{grad}_0)$ and target space $L_2(\Omega)^N$. Hence, A as an operator acting on $L_2(\Omega) \oplus R(\text{grad}_0) \oplus N(\text{div})$ may be written as follows

$$A = \begin{pmatrix} 0 & \text{div} P^* & 0 \\ P \text{grad}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Observe that A leaves the space $L_2(\Omega) \oplus R(\text{grad}_0)$ invariant. Moreover, note that A is one-to-one and skew-selfadjoint on $L_2(\Omega) \oplus R(\text{grad}_0)$. Furthermore, it is not hard too see

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that the domain of $\operatorname{div} P^*$ if endowed with the graph norm is compactly embedded into $R(\operatorname{grad}_0)$, see e.g. [31, Lemma 4.1]. Thus, the operator $\begin{pmatrix} 0 & \operatorname{div} P^* \\ P \operatorname{grad}_0 & 0 \end{pmatrix}$ has compact resolvent.

Denoting by $Q: L_2(\Omega)^N \rightarrow N(\operatorname{div})$ the operator, which maps $g \in L_2(\Omega)^N$ to its orthogonal projection $Qg \in N(\operatorname{div})$, we deduce from (1.5) the following system

$$\left(\partial_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & P\kappa(n\cdot)^{-1}P^* & P\kappa(n\cdot)^{-1}Q^* \\ 0 & Q\kappa(n\cdot)^{-1}P^* & Q\kappa(n\cdot)^{-1}Q^* \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} P^* & 0 \\ P \operatorname{grad}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \theta_n \\ q_n^1 \\ q_n^2 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}. \quad (1.6)$$

Now, we could apply Theorem 4.3 to the first two rows of equation (1.6) and write the term $P\kappa(n\cdot)^{-1}Q^*q_n^2$ to the right-hand side, i.e.,

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & P\kappa(n\cdot)^{-1}P^* \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} P^* \\ P \operatorname{grad}_0 & 0 \end{pmatrix} \right) \begin{pmatrix} \theta_n \\ q_n^1 \end{pmatrix} = \begin{pmatrix} f \\ -P\kappa(n\cdot)^{-1}Q^*q_n^2 \end{pmatrix}.$$

A suitable choice of subsequences would then yield the weak convergence of $P\kappa(n\cdot)^{-1}Q^*q_n^2$. In order to get a similar equation as in the original system after letting $n \rightarrow \infty$, we need the limit of $P\kappa(n\cdot)^{-1}Q^*q_n^2$ to be of the form Cq^2 for a suitable bounded linear operator C and q^2 being the limit of $(q_n^2)_n$. The reason for that is that the limiting equation should contain the limit of q_n^2 as an unknown. However, in general, we cannot deduce that the limit of $P\kappa(n\cdot)^{-1}Q^*q_n^2$ is the product of the limits of $P\kappa(n\cdot)^{-1}Q^*$ and q_n^2 . A deeper look into Theorem 4.3 reveals that due to the compactness of the resolvent of $\begin{pmatrix} 0 & \operatorname{div} P^* \\ P \operatorname{grad}_0 & 0 \end{pmatrix}$ the sequence $(\theta_n, q_n^1)_n$ converges in a stronger sense than only weakly such that a particular ‘weak-strong principle’ can be applied, cf. Theorem 3.5. Hence, we want to express q_n^2 in terms of (θ_n, q_n^1) . Therefore we are led to perform Gauss steps to transform (1.6) into a system such that the convergence of $(q_n^2)_n$ can be deduced from the convergence of $(\theta_n, q_n^1)_n$. A possible way of doing so is the following

$$\left(\partial_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & P\kappa(n\cdot)^{-1}P^* - P\kappa(n\cdot)^{-1}Q^*(Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^* & 0 \\ 0 & Q\kappa(n\cdot)^{-1}P^* & Q\kappa(n\cdot)^{-1}Q^* \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} P^* & 0 \\ P \operatorname{grad}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \theta_n \\ q_n^1 \\ q_n^2 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}.$$

Multiplication by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (Q\kappa(n\cdot)^{-1}Q^*)^{-1} \end{pmatrix}$ gives

$$\begin{pmatrix} \partial_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & P\kappa(n\cdot)^{-1}P^* - P\kappa(n\cdot)^{-1}Q^*(Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^* & 0 \\ 0 & (Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^* & 1 \end{pmatrix} \\ + \begin{pmatrix} 0 & \operatorname{div} P^* & 0 \\ P \operatorname{grad}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \theta_n \\ q_n^1 \\ q_n^2 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}.$$

In the latter equation we may perform the limit at least for a suitable choice of subsequences, for which we use the same notation. To this end, note that the operators

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & P\kappa(n\cdot)^{-1}P^* - P\kappa(n\cdot)^{-1}Q^*(Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^* & 0 \\ 0 & (Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^* & 1 \end{pmatrix} \quad (n \in \mathbb{N})$$

form a bounded sequence in the space of linear operators in the separable Hilbert space $L_2(\Omega)^{N+1}$. Thus, there exists a subsequence, which converges in the weak operator topology of $L(L_2(\Omega)^{N+1})$. Applying Theorem 4.3 to the first two rows of the latter equation, i.e.,

$$\begin{pmatrix} \partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & P\kappa(n\cdot)^{-1}P^* - P\kappa(n\cdot)^{-1}Q^*(Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^* \end{pmatrix} \\ + \begin{pmatrix} 0 & \operatorname{div} P^* \\ P \operatorname{grad}_0 & 0 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \theta_n \\ q_n^1 \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix},$$

we deduce that $(\theta_n, q_n^1)_n$ converges weakly and due to the compact resolvent of the operator $\begin{pmatrix} 0 & \operatorname{div} P^* \\ P \operatorname{grad}_0 & 0 \end{pmatrix}$ the sequence $(\partial_0^{-3}\theta_n(t), \partial_0^{-3}q_n^1(t))_n$ converges strongly in $L_2(\Omega) \oplus R(\operatorname{grad}_0)$ for all $t \in \mathbb{R}$, cf. Theorem 4.3. In particular, this means that

$$\left((Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^*q_n^1 \right)_n$$

converges to the product of the limits of $\left((Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^* \right)_n$ and $(q_n^1)_n$, see also Corollary 3.6 below. Hence, in the last row equation

$$(Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^*q_n^1 + q_n^2 = 0$$

it is possible to let n tend to infinity. We get that $(\theta_n, q_n^1, q_n^2)_n$ weakly converges to a solution of the following equation

$$\begin{pmatrix} \partial_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lim_{n \rightarrow \infty} \left(P\kappa(n\cdot)^{-1}P^* - P\kappa(n\cdot)^{-1}Q^*(Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^* \right) & 0 \\ 0 & \lim_{n \rightarrow \infty} \left((Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^* \right) & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} \theta \\ q^1 \\ q^2 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}$$

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$$+ \begin{pmatrix} 0 & \operatorname{div} P^* & 0 \\ P \operatorname{grad}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta \\ q^1 \\ q^2 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}.$$

Note that the system is of a rather simple form, since the right-hand side contains two zeros. In general, we have a more difficult situation, which then also results in a more involved limiting equation.

How is it possible to see that the limiting equation is well-posed? This requires the performance of other Gauss steps and so the well-posedness is not easy to see. Following the strategy of Section 6, we end up with the following equations

$$\begin{aligned} & \left(\partial_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. \\ & + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lim_{n \rightarrow \infty} (P\kappa(n\cdot)^{-1}P^* - P\kappa(n\cdot)^{-1}Q^*(Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^*) & \lim_{n \rightarrow \infty} (P\kappa(n\cdot)^{-1}Q^*(Q\kappa(n\cdot)^{-1}Q^*)^{-1}) \\ 0 & \lim_{n \rightarrow \infty} ((Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^*) & \lim_{n \rightarrow \infty} (Q\kappa(n\cdot)^{-1}Q^*)^{-1} \end{pmatrix} \\ & \left. + \begin{pmatrix} 0 & \operatorname{div} P^* & 0 \\ P \operatorname{grad}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \begin{pmatrix} \theta \\ q^1 \\ q^2 \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ 0 \end{pmatrix}. \quad (1.7) \end{aligned}$$

The latter equation can be shown to be well-posed, time-translation invariant and to have a causal solution operator. Note that, if one is only interested in the behavior of the temperature distribution, we can reformulate the latter equation into a second order form. The resulting equation would be

$$\partial_0 \theta - \operatorname{div} P^* \left(\lim_{n \rightarrow \infty} P\kappa(n\cdot)^{-1}P^* - P\kappa(n\cdot)^{-1}Q^*(Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^* \right)^{-1} P \operatorname{grad}_0 \theta = f.$$

Using the periodicity of κ , we deduce that $P\kappa(n\cdot)^{-1}P^*$ converges in the weak operator topology to $P \int_{(0,1)^N} \kappa(x)^{-1} dx P^*$, cf. [31, Proposition 4.3]. Thus, in order to answer the question proposed in the title of this article, the (inverse of the) classical homogenized matrix and the harmonic mean of κ differ on the subspace $R(\operatorname{grad}_0)$ by the limit of the sequence $\left(P\kappa(n\cdot)^{-1}Q^*(Q\kappa(n\cdot)^{-1}Q^*)^{-1}Q\kappa(n\cdot)^{-1}P^* \right)_n$ in the weak operator topology.

The interested reader might think, why such a seemingly complicated strategy yielding the homogenized equations should be applied. In the case of the heat equation this strategy indeed does not give anything new despite the fact that the homogenized equations have a different representation. For the heat equation it is even worse: with this strategy one cannot deduce the convergence of the whole sequence. However, note that the approach presented here only uses abstract theory from functional analysis and does not rely on the specific form of κ being a periodic multiplication operator. If κ is a linear operator invoking non-local terms, well-known homogenization theory might fail to work. Moreover, the way of computing the homogenized coefficients carries over to a large class of evolutionary equations: It is possible to treat Maxwell's equations, the wave equation, the heat equation or general coupled systems in mathematical physics in a *unified* manner. It is also possible that a second order formulation might not be available or is not easy to handle, cf. e.g.

[22, Equation (6.3.9), p. 455] so that the usual strategy might not work.

Now, we give a detailed account of the rigorous treatment of evolutionary equations in the abstract setting of equations (0.1) and (0.2).

2 Setting

We recall the setting of evolutionary equations established in [21] or [22, Chapter 6]. Let H be a Hilbert space and denote by $L_2(\mathbb{R}; H)$ the space of H -valued L_2 -functions. The operator

$$\partial : W_2^1(\mathbb{R}; H) \subseteq L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H) : f \mapsto f'$$

assigning to each weakly differentiable H -valued function its weak derivative is skew-selfadjoint. Define the unitary Fourier transform $\mathcal{F} : L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H)$ as the closure of the mapping

$$f \mapsto \left(\xi \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) \, dx \right)$$

defined for f belonging to $C_c^\infty(\mathbb{R}; H)$ the set of indefinitely differentiable, H -valued functions with compact support. Let

$$m : \{f \in L_2(\mathbb{R}; H); (x \mapsto xf(x)) \in L_2(\mathbb{R}; H)\} \subseteq L_2(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H) : f \mapsto (x \mapsto xf(x)).$$

For $\nu > 0$ define $H_{\nu,0}(\mathbb{R}; H) := L_2(\mathbb{R}, \exp(-2\nu x) \, dx; H)$ the space of H -valued (equivalence classes of) square-integrable functions with respect to the weighted Lebesgue measure with Radon-Nikodym derivative $\exp(-2\nu(\cdot))$. We also write $H_{\nu,0}(\mathbb{R})$ if $H = \mathbb{C}$. The mapping $\exp(-\nu m) : H_{\nu,0}(\mathbb{R}; H) \rightarrow L_2(\mathbb{R}; H) : f \mapsto (x \mapsto \exp(-\nu x)f(x))$ is unitary and the operator

$$\partial_{0,\nu} := \exp(-\nu m)^*(\partial + \nu) \exp(-\nu m)$$

is normal in $H_{\nu,0}(\mathbb{R}; H)$. If there is no risk of confusion, we simply write ∂_0 instead of $\partial_{0,\nu}$. We have $\partial_{0,\nu}^{-1} \in L(H_{\nu,0}(\mathbb{R}; H))$ with $\|\partial_{0,\nu}^{-1}\| = 1/\nu$. Introducing the *Fourier-Laplace transform* $\mathcal{L}_\nu := \mathcal{F} \exp(-\nu m)$, we get

$$\partial_{0,\nu} = \mathcal{L}_\nu^*(im + \nu)\mathcal{L}_\nu.$$

Consequently,

$$\partial_{0,\nu}^{-1} = \mathcal{L}_\nu^*(im + \nu)^{-1} \mathcal{L}_\nu.$$

The latter equation gives a functional calculus for the normal operator $\partial_{0,\nu}^{-1}$:

Definition (Hardy space and functional calculus for $\partial_{0,\nu}$). For an open set $E \subseteq \mathbb{C}$ and a Banach space X , we define the Hardy space

$$\mathcal{H}^\infty(E; X) := \{M : E \rightarrow X; M \text{ bounded, analytic}\}$$

2 Setting

and $\|M\|_\infty := \sup\{|M(z)|_X; z \in E\}$. Let H_1, H_2 be Hilbert spaces, $\nu > 0$, $r > 1/(2\nu)$.

- For $M \in \mathcal{H}^\infty(B_\mathbb{C}(r, r); L(H_1, H_2))$ define

$$M(\partial_{0,\nu}^{-1}) := \mathcal{L}_\nu^* \left(M \left(\frac{1}{im + \nu} \right) \right) \mathcal{L}_\nu,$$

where $(M(\frac{1}{im+\nu})\phi)(t) = M(\frac{1}{it+\nu})\phi(t)$ for all $t \in \mathbb{R}$ and $\phi \in C_c^\infty(\mathbb{R}; H_1)$.

- For $c > 0$ define

$$\mathcal{H}^{\infty,c}(B_\mathbb{C}(r, r); L(H_1)) := \{M \in \mathcal{H}^\infty(B_\mathbb{C}(r, r); L(H_1)); \operatorname{Re} z^{-1} M(z) \geq c \ (z \in B_\mathbb{C}(r, r))\}.$$

For easy reference, we call elements of $\mathcal{H}^\infty(B_\mathbb{C}(r, r); L(H_1))$ *material laws* or *constitutive relations* and elements of $\mathcal{H}^{\infty,c}(B_\mathbb{C}(r, r); L(H_1))$ *(c)-material laws*.

As the definition of ∂_0 indicates, we will deal with space-time operators. In particular, we identify any closed, densely defined operator $A : D(A) \subseteq H \rightarrow H$ in some Hilbert space H with its canonical extension on the space of H -valued $H_{\nu,0}(\mathbb{R})$ -functions, cf. [21]. We have the following well-posedness theorem taken from [21].

Theorem 2.1 ([21, Solution theory]). *Let H be a Hilbert space, $c, \nu > 0, r > 1/(2\nu)$ and $M \in \mathcal{H}^{\infty,c}(B(r, r); L(H))$. Let $A : D(A) \subseteq H \rightarrow H$ be skew-selfadjoint. Then the equation*

$$(\partial_0 M(\partial_0^{-1}) + A)u = f$$

admits a unique solution $u \in H_{\nu,0}(\mathbb{R}; H)$ for a dense set of right-hand sides $f \in H_{\nu,0}(\mathbb{R}; H)$. Moreover, the solution operator $(\partial_0 M(\partial_0^{-1}) + A)^{-1}$ is a densely defined, continuous operator in $H_{\nu,0}(\mathbb{R}; H)$ with operator norm bounded by $\frac{1}{c}$, and the operator $(\partial_0 M(\partial_0^{-1}) + A)^{-1}$ is causal, i.e., for all $f \in H_{\nu,0}(\mathbb{R}; H)$ and $a \in \mathbb{R}$ we have

$$\chi_{(-\infty, a)}(m) \overline{(\partial_0 M(\partial_0^{-1}) + A)^{-1}(\chi_{(-\infty, a)}(m)f)} = \chi_{(-\infty, a)}(m) \overline{(\partial_0 M(\partial_0^{-1}) + A)^{-1}(f)},$$

where $\chi_{(-\infty, a)}(m)$ denotes the multiplication operator mapping $f \in H_{\nu,0}(\mathbb{R}; H)$ to the truncated function $t \mapsto \chi_{(-\infty, a)}(t)f(t)$.

We note that these results carry over to “tailor made” distribution spaces – so-called Sobolev lattices – discussed in [20]. In [31, Remarks 1.2: (i)–(iii)] and [21, Sections 2 and 3] the core issues are sketched. We will use the notation from [31] and for the sake of clarity, we recall the main definitions. For $k \in \mathbb{Z}$, a Hilbert space H and a densely defined, closed linear operator $C : D(C) \subseteq H \rightarrow H$ with $0 \in \varrho(C)$, we denote by $H_k(C)$ the Hilbert space defined as the completion of $D(C^{|k|})$ with respect to the norm $|\cdot|_{H_k(C)} : u \mapsto |C^k u|_H$. It can be shown that the closure of $H_{|k|}(C) \subseteq H_k(C) \rightarrow H_{k-1}(C) : u \mapsto Cu$ is unitary. We will re-use the name C for this extension. We are interested in the special cases $C = A + 1$ with A skew-selfadjoint or $C = \partial_0$. For $\ell \in \{-1, 0, 1\}$ we let $H_{\ell,A} := H_\ell(A + 1)$. For ∂_0 defined

on $H_{\nu,0}(\mathbb{R})$ -functions with values in a Hilbert space H we write $H_{\nu,k}(\mathbb{R}; H) := H_k(\partial_0)$. Consequently, we also use the spaces $H_{\nu,k}(\mathbb{R}; H_{\ell,A})$, $\ell \in \{-1, 0, 1\}$. The extension of the solution operator to $H_{\nu,-1}(\mathbb{R}; H)$ also serves as a way to model initial value problems, see e.g. [22, 6.2.5 Initial Value Problems].

3 Preliminary results

We summarize some findings from [30, 31].

Definition. For an open set $E \subseteq \mathbb{C}$ and Hilbert spaces H_1, H_2 , we define on the set $\mathcal{H}^\infty(E; L(H_1, H_2))$ the initial topology $\tau_{\mathcal{M}}$ induced by the mappings

$$\mathcal{H}^\infty(E; L(H_1, H_2)) \ni M \mapsto \langle \phi, M(\cdot)\psi \rangle \in \mathcal{H}(E),$$

where $\mathcal{H}(E)$ is the set of holomorphic \mathbb{C} -valued functions endowed with the compact open topology. We define $\mathcal{H}_w^\infty(E; L(H_1, H_2)) := (\mathcal{H}^\infty(E; L(H_1, H_2)), \tau_{\mathcal{M}})$ and re-use the name $\mathcal{H}_w^\infty(E; L(H_1, H_2))$ for the underlying set.

Theorem 3.1 (sequential compactness, [30, Theorem 3.4]). *Let H_1, H_2 be separable Hilbert spaces, $E \subseteq \mathbb{C}$ open. Let $\mathcal{B} \subseteq \mathcal{H}_w^\infty(E; L(H_1, H_2))$ be bounded, i.e., $\sup\{\|M(z)\|; z \in E, M \in \mathcal{B}\} < \infty$. Then \mathcal{B} is relatively sequentially compact.*

Lemma 3.2 ([30, Lemma 3.5]). *Let H be a Hilbert space, $r > 0$. Let $(M_n)_n$ be a bounded and convergent sequence in the space $\mathcal{H}_w^\infty(B(r, r); L(H_1, H_2))$ with limit $M \in \mathcal{H}_w^\infty(B(r, r); L(H_1, H_2))$. Then $(M_n(\partial_0^{-1}))_n$ converges to $M(\partial_0^{-1})$ in the weak operator topology of $L(H_{\nu,k}(\mathbb{R}; H_1), H_{\nu,k}(\mathbb{R}; H_2))$, where $\nu > 1/(2r)$, $k \in \mathbb{Z}$.*

Proof. In [30, Lemma 3.5], the claim was shown for the case $k = 0$ and $H_1 = H_2$. The general case follows by observing that $\partial_0^k : H_{\nu,k}(\mathbb{R}; H_1) \rightarrow H_{\nu,0}(\mathbb{R}; H_1)$ is unitary and obvious modifications. \square

Lemma 3.3 ([29, Lemma 1.5]). *Let H_1, H_2 be Hilbert spaces. Let $E \subseteq \mathbb{C}$ be an open disc with centre z and let $(M_n)_n = (\sum_{k=0}^\infty (\cdot - z)^k A_{nk})_n$ be a convergent sequence in $\mathcal{H}_w^\infty(E; L(H_1, H_2))$ with limit $\sum_{k=0}^\infty (\cdot - z)^k A_k$. Then $A_{nk} \rightarrow A_k$ as $n \rightarrow \infty$ in the weak operator topology of $L(H_1, H_2)$ for all $k \in \mathbb{N}_0$.*

For a Hilbert space H and $\nu > 0$, we define

$$C_\nu(\mathbb{R}; H) := \{\phi \in C(\mathbb{R}; H); \sup_{t \in \mathbb{R}} |\exp(-\nu t)\phi(t)|_H < \infty\}.$$

We endow $C_\nu(\mathbb{R}; H)$ with the norm $|\cdot|_{C_\nu} : \phi \mapsto \sup_{t \in \mathbb{R}} |\exp(-\nu t)\phi(t)|_H$. Recall from [22, Lemma 3.1.59] that $H_{\nu,1}(\mathbb{R}; H)$ is continuously embedded in $C_\nu(\mathbb{R}; H)$.

3 Preliminary results

Lemma 3.4 ([31, Lemma 2.2]). *Let H be a Hilbert space, $\nu > 0$. If $(f_n)_n$ in $H_{\nu,1}(\mathbb{R}; H)$ is bounded and converges pointwise to some $f \in H_{\nu,1}(\mathbb{R}; H)$, then*

$$\partial_0^{-1} f_n(t) \xrightarrow{n \rightarrow \infty} \partial_0^{-1} f(t),$$

for all $t \in \mathbb{R}$.

Theorem 3.5 (weak-strong principle, [31, Theorem 2.3]). *Let H be a Hilbert space, $\varepsilon > 0$, $(M_n)_n$ be a convergent sequence in $\mathcal{H}_w^\infty(B_{\mathbb{C}}(0, \varepsilon); L(H_1, H_2))$ with limit M . Then, for $\nu > 2/\varepsilon$ and any bounded sequence $(v_n)_n$ in $H_{\nu,1}(\mathbb{R}; H_1)$ and $v \in H_{\nu,1}(\mathbb{R}; H_1)$ such that $v_n(t) \xrightarrow{n \rightarrow \infty} v(t)$ in H_1 for all $t \in \mathbb{R}$,*

$$\text{w-} \lim_{n \rightarrow \infty} (M_n(\partial_0^{-1} v_n)(t) = (M(\partial_0^{-1} v)(t) \in H_2,$$

for all $t \in \mathbb{R}$.

Proof. In [31] the proof is given for the case $H_1 = H_2$. The assertion follows analogously with obvious modifications. \square

Corollary 3.6. *Let H_1, H_2 be Hilbert spaces, $\varepsilon > 0$, $(M_n)_n$ be a convergent sequence in $\mathcal{H}_w^\infty(B_{\mathbb{C}}(0, \varepsilon); L(H_1, H_2))$ with limit $M \in \mathcal{H}_w^\infty(B_{\mathbb{C}}(0, \varepsilon); L(H_1, H_2))$. Let $\nu > 2/\varepsilon$, $k \in \mathbb{Z}$ and let $(v_n)_n$ be bounded in $H_{\nu,k}(\mathbb{R}; H_1)$, $v \in H_{\nu,k}(\mathbb{R}; H_1)$. Assume there is $l \in \mathbb{N}_0$ such that $\partial_0^{-l} v_n \in H_{\nu,1}(\mathbb{R}; H_1)$ and $\partial_0^{-l} v_n(t) \xrightarrow{n \rightarrow \infty} \partial_0^{-l} v(t)$ in H_1 for all $t \in \mathbb{R}$. Then*

$$\text{w-} \lim_{n \rightarrow \infty} M_n(\partial_0^{-1} v_n) = M(\partial_0^{-1} v) \in H_{\nu,k}(\mathbb{R}; H_2).$$

Proof. Since $(M_n(\partial_0^{-1} v_n))_n$ is bounded in $H_{\nu,k}(\mathbb{R}; H_2)$, there is a subsequence with indices $(n_j)_j$ weakly converging to some $w \in H_{\nu,k}(\mathbb{R}; H_2)$. The assumption guarantees that $(\partial_0^{-|k|-l} v_n)_n$ is bounded in $H_{\nu,1}(\mathbb{R}; H_1)$. Moreover, by Lemma 3.4, $(\partial_0^{-|k|-l} v_n)_n$ converges pointwise to $\partial_0^{-|k|-l} v$. Thus, by Theorem 3.5 and the weak continuity of point-evaluation, we deduce that, for $t \in \mathbb{R}$,

$$\begin{aligned} (\partial_0^{-|k|-l} w)(t) &= \text{w-} \lim_{j \rightarrow \infty} (\partial_0^{-|k|-l} M_{n_j}(\partial_0^{-1} v_{n_j}))(t) \\ &= \text{w-} \lim_{j \rightarrow \infty} M_{n_j}(\partial_0^{-1} \partial_0^{-|k|-l} v_{n_j})(t) \\ &= M(\partial_0^{-1} \partial_0^{-|k|-l} v)(t) = \partial_0^{-|k|-l} M(\partial_0^{-1} v)(t). \end{aligned}$$

Hence, $w = M(\partial_0^{-1} v)$. \square

4 A general compactness theorem for the homogenization of evolutionary equations

We introduce the concept of G -convergence to bridge the gap between the classical approach in homogenization theory and the Hilbert space perspective discussed here.

Definition (G -convergence, [36, p. 74]). Let H be a Hilbert space. Let $(A_n : D(A_n) \subseteq H \rightarrow H)_n$ be a sequence of one-to-one mappings onto H and let $B : D(B) \subseteq H \rightarrow H$ be one-to-one. We say that $(A_n)_n$ G -converges to B if for all $f \in H$ the sequence $(A_n^{-1}(f))_n$ converges weakly to some u , which satisfies $u \in D(B)$ and $B(u) = f$. B is called a G -limit of $(A_n)_n$. We say that $(A_n)_n$ strongly G -converges to B in H , if for all weakly converging sequences $(f_n)_n$ in H , $(A_n^{-1}(f_n))_n$ weakly converges to some u , which satisfies $u \in D(B)$ and $B(u) = \text{w-}\lim_{n \rightarrow \infty} f_n$.

Proposition 4.1. *The G -limit is uniquely determined.*

Proof. Let H be a Hilbert space. Let $(A_n)_n$ be a sequence of one-to-one onto mappings which is G -convergent to the one-to-one mapping $B : D(B) \subseteq H \rightarrow H$. Define $C := \{(u, f) \in H \oplus H; u = \text{w-}\lim_{n \rightarrow \infty} A_n^{-1}(f)\}$. Then $C \subseteq B$, so that C is a mapping. Moreover, since C is onto and B is one-to-one, we conclude that $C = B$. \square

Remark 4.2. If $(A_n)_n$ in the above definition is in addition a sequence of linear and closed operators and B is also closed and linear, then the above definition of G -convergence is precisely convergence of the resolvents in the weak operator topology, which is the original definition in [36] in the Hilbert space setting.

We now prove compactness results concerning G -convergence for operators that may be associated with evolutionary equations. More precisely, we will deal with the following cases:

Definition. Let H_1, H_2 be Hilbert spaces. We say a pair $((M_n)_n, \mathcal{A})$ satisfies

- (P1) if there exists $\varepsilon, r, c > 0$ such that $(M_n)_n$ is a bounded sequence in $\mathcal{H}^\infty(B(0, \varepsilon); L(H_1)) \cap \mathcal{H}^{\infty, c}(B(r, r); L(H_1))$ and $\mathcal{A} : D(\mathcal{A}) \subseteq H_1 \rightarrow H_1$ is skew-selfadjoint and the embedding $(D(\mathcal{A}), |\cdot|_{\mathcal{A}}) \hookrightarrow (H_1, |\cdot|_{H_1})$ is compact,
- (P2) if there exists $\varepsilon, c, r > 0$ such that $(M_n)_n = \left(\begin{pmatrix} M_{11,n} & M_{12,n} \\ M_{21,n} & M_{22,n} \end{pmatrix} \right)_n$ is bounded in $\mathcal{H}^\infty(B(0, \varepsilon); L(H_1 \oplus H_2)) \cap \mathcal{H}^{\infty, c}(B(r, r); L(H_1 \oplus H_2))$ and $\mathcal{A} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ is such that $((M_{11,n})_n, A)$ satisfies (P1). Moreover,
 - (i) for all $n \in \mathbb{N}$, $R(M_1(0)) = R(M_n(0))$ and $M_n(0) \geq c$ on $R(M_1(0))$,

4 A general compactness theorem for the homogenization of evolutionary equations

(ii) denoting by $q_j : H_j \rightarrow R(\pi_j^*) \cap N(M_1(0))$ ($j \in \{1, 2\}$) the canonical orthogonal projections, we have for all $n \in \mathbb{N}$

$$\left((q_2 M'_{22,n}(0) q_2^*)^{-1} q_2 M'_{21,n}(0) q_1^* \right)^* = q_1 M'_{12,n}(0) q_2^* (q_2 M'_{22,n}(0) q_2^*)^{-1}.$$

With these definitions, the core result in [31] now reads as follows.

Theorem 4.3 ([31, Theorem 3.5]). *Let H be a Hilbert space and assume that $((M_n)_n, \mathcal{A})$ satisfies (P1) and that $(M_n)_n$ converges to $N \in \mathcal{H}_w^\infty(B(0, \varepsilon); L(H))$. Then there exists $\nu_0 \geq 0$ such that for all $\nu > \nu_0$, $(\partial_0 M_{n_k}(\partial_0^{-1}) + \mathcal{A})_k$ strongly G -converges to $\partial_0 N(\partial_0^{-1}) + \mathcal{A}$ in $H_{\nu, -1}(\mathbb{R}; H)$. Moreover, $N \in \mathcal{H}^{\infty, c}(B(r, r); L(H))$ and*

$$\partial_0^{-3}(\partial_0 M_n(\partial_0^{-1}) + \mathcal{A})^{-1} f_n(t) \rightarrow \partial_0^{-3}(\partial_0 N(\partial_0^{-1}) + \mathcal{A})^{-1} (\text{w-} \lim_{n \rightarrow \infty} f_n)(t) \in H$$

as $n \rightarrow \infty$ for all $t \in \mathbb{R}$ and all weakly convergent sequences $(f_n)_n$ in $H_{\nu, -1}(\mathbb{R}; H)$.

The generalization of this theorem to the case (P2) requires a homogenization result for the case of $A = 0$, i.e. a result on the homogenization of ordinary integro-differential equations. Since we deal with a possibly degenerated case in the sense of [21, 3.3 Some special cases], we cannot use the homogenization result for ordinary integro-differential equations already established in [30, Theorem 5.2]. The refined argument is tailored for the 0-analytic case (cf. Section 6), which, however, does not cover the results in [30].

Theorem 4.4. *Let H be a separable Hilbert space, $\varepsilon, c, d, r > 0$. Let $(M_n)_n$ be a bounded sequence in $\mathcal{H}^\infty(B(0, \varepsilon); L(H)) \cap \mathcal{H}^{\infty, c}(B(r, r); L(H))$ and assume that for all $n \in \mathbb{N}$, $M_n(0) \geq d$ on $R(M_n(0)) = R(M_1(0))$. Then there exists $r' \in (0, r]$ and a strictly monotone sequence of positive integers $(n_k)_k$ such that, for $\nu > 1/(2r')$, $(\partial_0 M_{n_k}(\partial_0^{-1}))_k$ G -converges to $\partial_0 \mu(\partial_0^{-1})$ in $H_{\nu, -1}(\mathbb{R}; H)$, where μ has the following properties: there is $\varepsilon', c' > 0$ such that*

- (i) $\mu \in \mathcal{H}^\infty(B(0, \varepsilon'); L(H)) \cap \mathcal{H}^{\infty, c'}(B(r', r'); L(H))$,
- (ii) $R(\mu(0)) = R(M_1(0))$,
- (iii) for all open $E \subseteq \mathbb{C}$ relatively compact in $B(0, \varepsilon') \setminus \{0\}$ ($\Leftrightarrow E \subset\subset B(0, \varepsilon') \setminus \{0\}$),

$$M_{n_k}(\cdot)^{-1} \rightarrow \mu(\cdot)^{-1} \in \mathcal{H}_w^\infty(E; L(H)) \quad (k \rightarrow \infty).$$

Proof. Define the Hilbert spaces $H_1 := R(M_1(0))$ and $H_2 := N(M_1(0))$ together with the canonical (orthogonal) projections $\pi_j : H \rightarrow H_j$, $j \in \{1, 2\}$. Then, for all $n \in \mathbb{N}$ and $j, k \in \{1, 2\}$, set $M_{jk,n}(\cdot) := \pi_j M_n(\cdot) \pi_k^*$. Now, the first assertion in Lemma 6.10 ensures the existence of $\varepsilon' > 0$ such that, for all $E \subseteq \mathbb{C}$ relatively compact in $B(0, \varepsilon') \setminus \{0\}$, the sequence $(M_n(\cdot)^{-1})_n$ is bounded in $\mathcal{H}^\infty(E; L(H_1 \oplus H_2))$. By σ -compactness of $B(0, \varepsilon') \setminus \{0\}$ and Theorem 3.1, we may choose a subsequence $(M_{n_k}(\cdot)^{-1})_k$ of $(M_n(\cdot)^{-1})_n$ such that there

is a holomorphic mapping $\eta : B(0, \varepsilon') \setminus \{0\} \rightarrow L(H)$ with

$$M_{n_k}(\cdot)^{-1} \rightarrow \eta \in \mathcal{H}_w^\infty(E; L(H)) \quad (k \rightarrow \infty, E \subset\subset B(0, \varepsilon') \setminus \{0\}).$$

The residue theorem ensures that the residues of $M_{n_k}(\cdot)^{-1}$ converge to the one of η in the weak operator topology. Moreover, it is easy to see that the coefficients of the Laurent series expansions of $M_{n_k}(\cdot)^{-1}$ converge in the weak operator topology τ_w . With the help of the first assertion of Lemma 6.10, the Laurent series expansion of η is of the form

$$\eta(z) = \begin{pmatrix} (\tau_w^-) \lim_{k \rightarrow \infty} M_{11, n_k}(0)^{-1} + \widehat{M}_{11}(z) & \widehat{M}_{12}(z) \\ \widehat{M}_{21}(z) & z^{-1}(\tau_w^-) \lim_{k \rightarrow \infty} M_{22, n_k}(0)^{-1} + \widehat{M}_{22}(z) \end{pmatrix}$$

for suitable bounded holomorphic operator-valued functions \widehat{M}_{jk} for $j, k \in \{1, 2\}$. The second assertion of Lemma 6.10 yields the existence of $\varepsilon'' > 0$ such that $\mu := \eta(\cdot)^{-1} \in \mathcal{H}^\infty(B(0, \varepsilon''); L(H))$. Moreover, from the representation in Lemma 6.10, we read off that $R(M_1(0)) = R(\mu(0))$ and $\mu(0) \geq d'$ on H_1 for some $d' > 0$ according to Inequality (6.3) and the fact that positive definiteness is preserved under limits in the weak operator topology. Similarly, $\operatorname{Re} \mu'(0) \geq c' > 0$ on H_2 . Thus, by Remark 6.3 it follows that μ lies in $\mathcal{H}^{\infty, c''}(B(r', r''); L(H))$ for some $r', c'' > 0$. It remains to show the G -convergence result. To this end let $\nu > 1/(2r')$. By the convergence of the coefficients in the Laurent series of $(M_{n_k}(\cdot)^{-1})_k$, we get that $((\cdot)M_{n_k}(\cdot)^{-1})_k$ converges to $(\cdot)\eta(\cdot)$ in $\mathcal{H}_w^\infty(B(1/(2\nu), 1/(2\nu)); L(H))$. Thus, Lemma 3.2 implies that $((\partial_0 M_{n_k}(\partial_0^{-1}))^{-1})_k$ converges to $\partial_0^{-1}\eta(\partial_0^{-1})$ in the weak operator topology of $L(H_{\nu, -1}(\mathbb{R}; H))$. Employing Remark 4.2, we obtain the desired G -convergence. \square

Theorem 4.5. *Let H_1, H_2 be separable Hilbert spaces. Assume that $((M_n)_n, \mathcal{A})$ satisfies (P2). Then there exists $\nu_0 \geq 0, \varepsilon', c' > 0$ and $(n_k)_k$ a strictly monotone sequence of positive integers such that for all $\nu > \nu_0$ the sequence $(\partial_0 M_{n_k}(\partial_0^{-1}) + \mathcal{A})_k$ G -converges to $(\partial_0 N(\partial_0^{-1}) + \mathcal{A})$ in $H_{\nu, -1}(\mathbb{R}; H_1 \oplus H_2)$ with*

$$N(\cdot) := \begin{pmatrix} \eta_1(\cdot) + \eta_4(\cdot)\eta_2(\cdot)^{-1}\eta_3(\cdot) & \eta_4(\cdot)\eta_2(\cdot)^{-1} \\ \eta_2(\cdot)^{-1}\eta_3(\cdot) & \eta_2(\cdot)^{-1} \end{pmatrix} \\ \in \mathcal{H}^\infty(B(0, \varepsilon'); L(H_1 \oplus H_2)) \cap \mathcal{H}^{\infty, c'}(B(1/(2\nu_0), 1/(2\nu_0)); L(H_1 \oplus H_2)),$$

where

$$\begin{aligned} \eta_1(\cdot) &:= \lim_{k \rightarrow \infty} M_{11, n_k}(\cdot) - M_{12, n_k}(\cdot)M_{22, n_k}(\cdot)^{-1}M_{21, n_k}(\cdot) \in \mathcal{H}_w^\infty(B(0, \varepsilon'); L(H_1)) \\ \eta_2(\cdot) &:= \lim_{k \rightarrow \infty} (M_{22, n_k}(\cdot))^{-1} \in \mathcal{H}_w^\infty(E; L(H_2)) \quad (E \subset\subset B(0, \varepsilon') \setminus \{0\}) \\ \eta_3(\cdot) &:= \lim_{k \rightarrow \infty} M_{22, n_k}(\cdot)^{-1}M_{21, n_k}(\cdot) \in \mathcal{H}_w^\infty(B(0, \varepsilon'); L(H_1, H_2)) \text{ and} \\ \eta_4(\cdot) &:= \lim_{k \rightarrow \infty} M_{12, n_k}(\cdot)M_{22, n_k}(\cdot)^{-1} \in \mathcal{H}_w^\infty(B(0, \varepsilon'); L(H_2, H_1)). \end{aligned}$$

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Moreover, $R(N(0)) = R(M_1(0))$.

Proof. By Theorem 6.11 (applied to $M = M_n$ and the sequence N just the constant sequence consisting of M_n as every entry) there exist $\varepsilon', r', c' > 0$ such that, for all $n \in \mathbb{N}$,

$$\begin{aligned} & \begin{pmatrix} 1 - M_{12,n}(\cdot)M_{22,n}(\cdot)^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_{11,n}(\cdot) & M_{12,n}(\cdot) \\ M_{21,n}(\cdot) & M_{22,n}(\cdot) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -M_{22,n}(\cdot)^{-1}M_{21,n}(\cdot) & 1 \end{pmatrix} \\ &= \begin{pmatrix} M_{11,n}(\cdot) - M_{12,n}(\cdot)M_{22,n}(\cdot)^{-1}M_{21,n}(\cdot) & 0 \\ 0 & M_{22,n}(\cdot) \end{pmatrix} \\ &\in \mathcal{H}^\infty(B(0, \varepsilon'); L(H_1 \oplus H_2)) \cap \mathcal{H}^{\infty, c'}(B(r', r'); L(H_1 \oplus H_2)). \end{aligned}$$

Let $\nu > 1/(2r')$. By Theorem 3.1 and Theorem 4.4, we may choose convergent subsequences of the material law sequences

$$\begin{aligned} (\mu_{1,n})_n &:= (M_{11,n}(\cdot) - M_{12,n}(\cdot)M_{22,n}(\cdot)^{-1}M_{21,n}(\cdot))_n \\ (\mu_{2,n})_n &:= (M_{22,n}(\cdot)^{-1})_n \\ (\mu_{3,n})_n &:= (M_{22,n}(\cdot)^{-1}M_{21,n}(\cdot))_n \text{ and} \\ (\mu_{4,n})_n &:= (M_{12,n}(\cdot)M_{22,n}(\cdot)^{-1})_n. \end{aligned}$$

We will use the same index for the subsequences and denote the respective limits by η_1, η_2, η_3 and η_4 . Using the representation from Theorem 6.6, we get with the help of Theorem 6.11:

$$\begin{aligned} (G_1 \oplus \{0\}) \oplus (G_3 \oplus \{0\}) &= R \left(\begin{pmatrix} \begin{pmatrix} M_{11,n}^{(0)} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} M_{13,n}^{(0)} & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} M_{31,n}^{(0)} & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} M_{33,n}^{(0)} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \right) \\ &= R \left(\begin{pmatrix} M_{11,n}(0) & M_{12,n}(0) \\ M_{21,n}(0) & M_{22,n}(0) \end{pmatrix} \right) \\ &= R \left(\begin{pmatrix} M_{11,n}(0) - M_{12,n}(0)M_{22,n}(0)^{-1}M_{21,n}(0) & 0 \\ 0 & M_{22,n}(0) \end{pmatrix} \right) \\ &= R(M_{11,n}(0) - M_{12,n}(0)M_{22,n}(0)^{-1}M_{21,n}(0)) \oplus R(M_{22,n}(0)). \end{aligned}$$

Now, $M_{11,n}(0) - M_{12,n}(0)M_{22,n}(0)^{-1}M_{21,n}(0)$ is strictly positive on $G_1 = R(M_{11,1}(0))$ and $M_{22,n}(0)$ is strictly positive on $G_3 = R(M_{22,1}(0))$ uniformly in n . Hence, we deduce that $R(\eta_1(0)) = R(M_{11,1}(0))$ and, from Theorem 4.4, that $R(\eta_2(\cdot)^{-1}(0)) = R(M_{22,1}(0))$. Let $(f_1, f_2) \in H_{\nu,-1}(\mathbb{R}; H_1 \oplus H_2)$ and for $n \in \mathbb{N}$, let $(u_{1,n}, u_{2,n}) \in H_{\nu,-1}(\mathbb{R}; H_1 \oplus H_2)$ be the unique solution of

$$\partial_0 \begin{pmatrix} M_{11,n}(\partial_0^{-1}) & M_{12,n}(\partial_0^{-1}) \\ M_{21,n}(\partial_0^{-1}) & M_{22,n}(\partial_0^{-1}) \end{pmatrix} \begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Multiplying this equation by $\begin{pmatrix} 1 & -M_{12,n}(\partial_0^{-1})M_{22,n}(\partial_0^{-1})^{-1} \\ 0 & (\partial_0 M_{22,n}(\partial_0^{-1}))^{-1} \end{pmatrix}$, we obtain

$$\begin{pmatrix} \partial_0 \mu_{1,n}(\partial_0^{-1}) & 0 \\ \mu_{3,n}(\partial_0^{-1}) & 1 \end{pmatrix} \begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix} = \begin{pmatrix} f_1 - \mu_{4,n}(\partial_0^{-1})f_2 \\ \mu_{2,n}(\partial_0^{-1})\partial_0^{-1}f_2 \end{pmatrix}.$$

Thus,

$$\begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix} = \begin{pmatrix} (\partial_0 \mu_{1,n}(\partial_0^{-1}) + A)^{-1}(f_1 - \mu_{4,n}(\partial_0^{-1})f_2) \\ -\mu_{3,n}(\partial_0^{-1})u_{1,n} + \mu_{2,n}(\partial_0^{-1})\partial_0^{-1}f_2 \end{pmatrix}.$$

Lemma 3.2 ensures that $(\mu_{4,n}(\partial_0^{-1})f_2)_n$ weakly converges to $\eta_4(\partial_0^{-1})f_2$. Thus, by Theorem 4.3, we deduce that $(u_{1,n})_n$ weakly converges to $(\partial_0 \eta_1(\partial_0^{-1}) + A)^{-1}(f_1 - \eta_4(\partial_0^{-1})f_2) =: v_1$. Moreover, $(\partial_0^{-3}u_{1,n})_n$ converges pointwise to $\partial_0^{-3}v_1$. Using the equality

$$u_{2,n} = -\mu_{3,n}(\partial_0^{-1})u_{1,n} + \mu_{2,n}(\partial_0^{-1})\partial_0^{-1}f_2 \in H_{\nu,-1}(\mathbb{R}; H_2),$$

we deduce, with the help of Corollary 3.6 for the first term on the right hand side and Theorem 4.4 for the second term, that

$$u_{2,n} \rightharpoonup v_2 := -\eta_3(\partial_0^{-1})v_1 + \eta_2(\partial_0^{-1})\partial_0^{-1}f_2 \in H_{\nu,-1}(\mathbb{R}; H_2)$$

as $n \rightarrow \infty$. We arrive at the limit system

$$\begin{pmatrix} \partial_0 \eta_1(\partial_0^{-1}) & 0 \\ \eta_3(\partial_0^{-1}) & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} f_1 - \eta_4(\partial_0^{-1})f_2 \\ \eta_2(\partial_0^{-1})\partial_0^{-1}f_2 \end{pmatrix}.$$

Multiplying this equation by $\begin{pmatrix} 1 & \partial_0 \eta_4(\partial_0^{-1})\eta_2(\partial_0^{-1})^{-1} \\ 0 & \eta_2(\partial_0^{-1})^{-1}\partial_0 \end{pmatrix}$, we obtain

$$\partial_0 \begin{pmatrix} \eta_1(\partial_0^{-1}) + \eta_4(\partial_0^{-1})\eta_2(\partial_0^{-1})^{-1}\eta_3(\partial_0^{-1}) & \eta_4(\partial_0^{-1})\eta_2(\partial_0^{-1})^{-1} \\ \eta_2(\partial_0^{-1})^{-1}\eta_3(\partial_0^{-1}) & \eta_2(\partial_0^{-1})^{-1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Next, we consider the operator

$$N(\cdot) = \begin{pmatrix} \eta_1(\cdot) + \eta_4(\cdot)\eta_2(\cdot)^{-1}\eta_3(\cdot) & \eta_4(\cdot)\eta_2(\cdot)^{-1} \\ \eta_2(\cdot)^{-1}\eta_3(\cdot) & \eta_2(\cdot)^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \eta_4(\cdot) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1(\cdot) & 0 \\ 0 & \eta_2(\cdot)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \eta_3(\cdot) & 1 \end{pmatrix}.$$

By Theorem 4.4, we deduce that $\eta_2(\cdot)^{-1}$ is a (c'') -material law with strictly positive zeroth order term on the range of $M_{22,1}(0)$ for some $c'' > 0$. Moreover, η_1 is a (c') -material law by Theorem 4.3. Hence, using Theorem 6.11, we deduce the existence of $\varepsilon'', r'', c''' > 0$ such that $N \in \mathcal{H}^{\infty, c'''}(B(r'', r''); L(H_1 \oplus H_2)) \cap \mathcal{H}^{\infty}(B(0, \varepsilon''); L(H_1 \oplus H_2))$. \square

Remark 4.6. Theorem 4.5 contains a structural result concerning homogenization. We have proved that 0-analytic material laws lead to 0-analytic material laws after the homogenization process. Hence, it cannot be expected that the homogenized material law contains fractional derivatives with respect to time or explicit delay terms, e.g. a delay

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operator like $\tau_{-h}f := f(\cdot - h)$ for some $h > 0$, as these are not analytic at 0, see e.g. [22, pp. 448 (a),(c)]. By Theorem 2.1 we see that the limit equation is also well-posed and causal. The assertion concerning the range of the material law N may be interpreted as “the main physical phenomenon remains unchanged under the homogenization process”. Indeed, the difference between the wave equation and the heat equation written as a first order system as in [29, Example 1.4.6] or [32, Example 3.2] is the range of the zeroth order term in the material law. Due to the limiting process there might be higher order parts in the expansion of the material law, which result in memory effects.

Corollary 4.7. *Let H be a separable Hilbert space, $\varepsilon, c, r > 0$, $A : D(A) \subseteq H \rightarrow H$ skew-selfadjoint. Denote by $P : H \rightarrow N(A)^\perp$, $Q : H \rightarrow N(A)$ the orthogonal projections onto the respective spaces $N(A)^\perp$ and $N(A)$. Assume that the operator A has the (NC) -property, i.e., $(D(PAP^*), |\cdot|_{PAP^*}) \hookrightarrow (H, |\cdot|_H)$ is compact. Let $(M_n)_n$ be a bounded sequence in $\mathcal{H}^\infty(B(0, \varepsilon); L(H)) \cap \mathcal{H}^{\infty, c}(B(r, r); L(H))$ with $M_n(0) \geq c$ on $R(M_n(0)) = R(M_1(0))$ for all $n \in \mathbb{N}$. Denote by $q_2 : H \rightarrow N(M_1(0)) \cap N(A)^\perp$, $q_4 : H \rightarrow N(M_1(0)) \cap N(A)$ the canonical orthogonal projections and assume*

$$q_2 M'_n(0) q_4^* (q_4 M'_n(0) q_4^*)^{-1} = q_2 M'_n(0)^* q_4^* (q_4 M'_n(0)^* q_4^*)^{-1} \text{ for all } n \in \mathbb{N}. \quad (4.1)$$

Then there exists $\nu_0 \geq 0, \varepsilon', c' > 0$ and $(n_k)_k$ a strictly monotone sequence of positive integers such that for all, $\nu > \nu_0$, the sequence

$$(\partial_0 M_{n_k}(\partial_0^{-1}) + A)_k$$

G-converges to

$$\begin{aligned} \partial_0 (P^* \eta_1(\partial_0^{-1}) P + P^* \eta_4(\partial_0^{-1}) \eta_2(\partial_0^{-1})^{-1} \eta_3(\partial_0^{-1}) P + P^* \eta_4(\partial_0^{-1}) \eta_2(\partial_0^{-1})^{-1} Q \\ + Q^* \eta_2(\partial_0^{-1})^{-1} \eta_3(\partial_0^{-1}) P + Q^* \eta_2(\partial_0^{-1})^{-1} Q) + A \end{aligned}$$

in $H_{\nu, -1}(\mathbb{R}; H)$, where²

$$\begin{aligned} \eta_1(\cdot) &:= \lim_{k \rightarrow \infty} PM_{n_k}(\cdot) P^* - PM_{n_k}(\cdot) Q^* (QM_{n_k}(\cdot) Q^*)^{-1} QM_{n_k}(\cdot) P^* \\ \eta_2(\cdot) &:= \lim_{k \rightarrow \infty} (QM_{n_k}(\cdot) Q^*)^{-1} \\ \eta_3(\cdot) &:= \lim_{k \rightarrow \infty} (QM_{n_k}(\cdot) Q^*)^{-1} QM_{n_k}(\cdot) P^* \text{ and} \\ \eta_4(\cdot) &:= \lim_{k \rightarrow \infty} (PM_{n_k}(\cdot) Q^*) (QM_{n_k}(\cdot) Q^*)^{-1}. \end{aligned}$$

Proof. The assertion follows by applying Theorem 4.5 to

$$((M_n)_n, \mathcal{A}) = \left(\left(\begin{pmatrix} PM_n(\cdot) P^* & PM_n(\cdot) Q^* \\ QM_n(\cdot) P^* & QM_n(\cdot) Q^* \end{pmatrix}_n, \begin{pmatrix} PAP^* & 0 \\ 0 & 0 \end{pmatrix} \right) \right). \quad \square$$

²The limits are computed in the way similar to Theorem 4.5 with $H_1 = N(A)^\perp$ and $H_2 = N(A)$.

Remark 4.8. The compatibility condition (4.1) may be hard to check in applications. However, there are some situations in which the Condition (4.1) is trivially satisfied:

- A is one-to-one; then $N(A) = \{0\}$ and $q_4 = 0$.
- $R(M_1(0)) \supseteq N(A)^\perp$; then $N(M_1(0)) \cap N(A)^\perp = \{0\}$ and $q_2 = 0$.
- $M_1(0)$ is onto; then the preceding condition is satisfied. We remark here that this condition was imposed in [29, Theorem 2.3.14]. This condition corresponds to hyperbolic-type equations in applications.
- $M'_n(0) = M'_n(0)^*$; then $q_2 M'_n(0) q_4^* (q_4 M'_n(0) q_4^*)^{-1} = q_2 M'_n(0)^* q_4^* (q_4 M'_n(0)^* q_4^*)^{-1}$.

We do not yet know whether the compatibility condition is optimal. We can however give some examples to show that the other assumptions in (P2) are reasonable. The following example shows that without the requirement on A to have the (NC)-property the limit equation can differ from the expressions given in Theorem 4.5 or Corollary 4.7.

Example 4.9 (Compactness assumption does not hold). Let $\nu, \varepsilon > 0$. Consider the mapping $a : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$a(x) := \chi_{[0, \frac{1}{2})}(x - k) + 2 \chi_{[\frac{1}{2}, 1]}(x - k)$$

for all $x \in [k, k + 1)$, where $k \in \mathbb{Z}$. Define the corresponding multiplication operator in $L_2(\mathbb{R})$, i.e. for $\phi \in C_c^\infty(\mathbb{R})$, we define $a(n \cdot \widehat{m})\phi := (x \mapsto a(nx)\phi(x))$ for $n \in \mathbb{N}$. Note that $a(x + k) = a(x)$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Let $f \in H_{\nu,0}(\mathbb{R}; L_2(\mathbb{R}))$. We consider the evolutionary equation with $(M_n(\partial_0^{-1}))_n := (\partial_0^{-1} a(n \cdot \widehat{m}))_n$ and $A = i : L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R}) : \phi \mapsto i\phi$. Clearly, $N(A) = \{0\}$. By [30, Theorem 4.5] or [10, Theorem 1.5], we deduce that

$$M_n \rightarrow \left(z \mapsto z \int_0^1 a(x) \, dx \right) = \left(z \mapsto \frac{3}{2}z \right) \in \mathcal{H}_w^\infty(B(0, \varepsilon); L(L_2(\mathbb{R})))$$

as $n \rightarrow \infty$. If the assertion of Corollary 4.7 remains true in this case, then $(\partial_0 M_n(\partial_0^{-1}) + A)_n$ G -converges to $\frac{3}{2} + i$. For $n \in \mathbb{N}$, let $u_n \in H_{\nu,0}(\mathbb{R}; L_2(\mathbb{R}))$ be the unique solution of the equation

$$(\partial_0 M_n(\partial_0^{-1}) + A)u_n = (a(n \cdot \widehat{m}) + i)u_n = f. \quad (4.2)$$

Observe that by [10, Theorem 1.5]

$$u_n = (a(n \cdot \widehat{m}) + i)^{-1} f \rightarrow \left(\int_0^1 (a(x) + i)^{-1} \, dx \right) f =: u.$$

as $n \rightarrow \infty$. We integrate

$$\int_0^1 (a(x) + i)^{-1} \, dx = \frac{1}{2}(1 + i)^{-1} + \frac{1}{2}(2 + i)^{-1}.$$

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Inverting the latter equation yields

$$\left(\int_0^1 (a(x) + i)^{-1} dx \right)^{-1} = \left(\frac{1}{2}(1 + i)^{-1} + \frac{1}{2}(2 + i)^{-1} \right)^{-1} = \frac{18}{13} + \frac{14}{13}i.$$

Hence, u satisfies

$$\left(\frac{3}{2} + i \right) u = f \text{ and } \left(\frac{18}{13} + \frac{14}{13}i \right) u = f,$$

which of course is a contradiction.

In the next example, the uniform positive definiteness is violated.

Example 4.10 (Uniform positive definiteness condition does not hold). Let $H = \mathbb{C}$, $\nu > 0$ and, for $n \in \mathbb{N}$, let $M_n(\partial_0^{-1}) = \partial_0^{-1} \frac{1}{n}$, $A = 0$, $f \in H_{\nu,0}(\mathbb{R}) \setminus \{0\}$. For $n \in \mathbb{N}$, let $u_n \in H_{\nu,0}(\mathbb{R})$ be defined by

$$\partial_0 M_n(\partial_0^{-1}) u_n = \frac{1}{n} u_n = f.$$

Then $(u_n)_n$ is not relatively weakly compact and contains no weakly convergent subsequence.

In the final example, the range condition in (P2) is violated.

Example 4.11 (Range condition does not hold). Let H be an infinite-dimensional, separable Hilbert space. Let $(\phi_n)_n$ be a complete orthonormal system. For $n \in \mathbb{N}$ define $M_n(\partial_0^{-1}) := \langle \phi_n, \cdot \rangle \phi_n + \partial_0^{-1}(1 - \langle \phi_n, \cdot \rangle \phi_n)$. For the sequence $(M_n)_n$ the range condition in (P2) (applied with $A = 0$) is violated. Let $f \in H_{\nu,0}(\mathbb{R}; H)$, $\nu > 0$. For $n \in \mathbb{N}$, let $u_n \in H_{\nu,0}(\mathbb{R}; H)$ be such that

$$\partial_0 M_n(\partial_0^{-1}) u_n = \partial_0 \langle \phi_n, u_n \rangle \phi_n + u_n - \langle \phi_n, u_n \rangle \phi_n = f.$$

It is easy to see that $(u_n)_n$ is bounded. Take the inner product of the last equation with ϕ_m for some $m \in \mathbb{N}$. If $n \in \mathbb{N}$ is larger than m we arrive at

$$\langle u_n, \phi_m \rangle = \langle f, \phi_m \rangle,$$

and we deduce that $(\partial_0 M_n(\partial_0^{-1}))_n$ G -converges to $\partial_0 \partial_0^{-1} = 1$. This, however, does not yield a differential equation.

5 Applications

We demonstrate the applicability of our main theorem, to the mathematical models of some physical phenomena. For notational details, we refer to [29, pp. 34 and p. 98] or to [28, 3.2 Examples].

Thermodynamics

Let $\alpha, \beta \in \mathbb{R}$, $0 < \alpha < \beta$, $\Omega \subseteq \mathbb{R}^N$ open and bounded, $N \in \mathbb{N}$. Recall [8, Definition 4.11]:

$$M(\alpha, \beta, \Omega) := \{\kappa \in L_\infty(\Omega)^{N \times N}; \operatorname{Re} \langle \kappa(x)\xi, \xi \rangle \geq \alpha|\xi|^2, |\kappa(x)\xi| \leq \beta|\xi|^2, \xi \in \mathbb{R}^N, \text{ a.e. } x \in \Omega\}.$$

Let $M_{\text{self}}(\alpha, \beta, \Omega) := \{\kappa \in M(\alpha, \beta, \Omega); \kappa \text{ selfadjoint a.e.}\}$. For $\kappa \in M(\alpha, \beta, \Omega)$ denote by $\kappa(\hat{m})$ the associated multiplication operator in $L_2(\Omega)^N$. Let $(\kappa_n)_n$ be a sequence in $M_{\text{self}}(\alpha, \beta, \Omega)$. Recall from Section 1 that a first order formulation of the heat equation with Dirichlet boundary conditions in the context of evolutionary equations introduced in [21] is the following

$$\left(\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \kappa_n(\hat{m})^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix} \right) \begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

We want to apply Corollary 4.7 with

$$A = \begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}$$

and

$$(M_n)_n := \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \partial_0^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \kappa_n(\hat{m})^{-1} \end{pmatrix} \right)_n.$$

The compactness condition on the operator $\begin{pmatrix} 0 & \operatorname{div} \\ \operatorname{grad}_0 & 0 \end{pmatrix}$ has already been established e.g. in [29, Remark 3.2.2] or [31, the end of the proof of Theorem 4.3] and the sequence $(\kappa_n(\hat{m})^{-1})_n$ is a sequence of selfadjoint operators. Since $M_n(0) = M_1(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for all $n \in \mathbb{N}$, the range condition is satisfied. Thus Corollary 4.7 applies. The sequence $(M_n)_n$ could be replaced by some convolution terms. Moreover it should be noted that the case of not necessarily symmetric κ_n 's has been considered in [31]. There, however, a second-order formulation was used which, for more general material laws, may not be available, see also the last paragraph in Section 1. The homogenized equations are derived in Section 1, see equation (1.7).

Electromagnetism

Let $\Omega \subseteq \mathbb{R}^3$. The general form of Maxwell's equations in bi-anisotropic dissipative media used in [3] is

$$\left(\partial_0 \begin{pmatrix} \varepsilon & \gamma \\ \gamma^* & \mu \end{pmatrix} + \begin{pmatrix} \sigma_{11}^* & \sigma_{12}^* \\ \sigma_{21}^* & \sigma_{22}^* \end{pmatrix} + \begin{pmatrix} 0 & \text{curl} \\ -\text{curl}_0 & 0 \end{pmatrix} \right) \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} J \\ 0 \end{pmatrix}$$

with a (c) -material law (see Section 2 for a definition) $M(\partial_0^{-1}) := \begin{pmatrix} \varepsilon & \gamma \\ \gamma^* & \mu \end{pmatrix} + \partial_0^{-1} \begin{pmatrix} \sigma_{11}^* & \sigma_{12}^* \\ \sigma_{21}^* & \sigma_{22}^* \end{pmatrix}$ for a $c > 0$. Here, σ_{jk} are $L(L_2(\Omega)^3)$ -valued functions on \mathbb{R} , vanishing on $\mathbb{R}_{<0}$ and being such that the temporal convolutions σ_{jk}^* yield 0-analytic material laws, $j, k \in \{1, 2\}$. Moreover, the operator $\begin{pmatrix} \varepsilon & \gamma \\ \gamma^* & \mu \end{pmatrix}$ is selfadjoint and strictly positive definite in $L_2(\Omega)^6$. We emphasize here that the convolution kernels may also take values in the linear operators on $L_2(\Omega)^3$, which are not representable as multiplication operators, thus, in this way, generalizing the assumptions in [3]. Now, consider a sequence of (c) -material laws $(M_n)_n$ of the above form with non-singular, strictly positive zeroth order term: $\begin{pmatrix} \varepsilon_n & \gamma_n \\ \gamma_n^* & \mu_n \end{pmatrix} \geq d > 0$ for all $n \in \mathbb{N}$. Then, Corollary 4.7 applies if we assume Ω to be bounded and to satisfy suitable smoothness assumptions on the boundary, see e.g. [19, 23, 35]. Indeed, the range condition is satisfied since $M_n(0) = \begin{pmatrix} \varepsilon_n & \gamma_n \\ \gamma_n^* & \mu_n \end{pmatrix}$ is onto for all $n \in \mathbb{N}$ and, since $N(M_n(0)) = \{0\}$, the compatibility condition (4.1) also follows. Note that the homogenized equations are more complicated than in the case of the heat equation. This is due to the fact that the orthogonal projection onto the nullspace of $A = \begin{pmatrix} 0 & \text{curl} \\ -\text{curl}_0 & 0 \end{pmatrix}$ acts in both components since both the operators curl and curl_0 have an (infinite-dimensional) nullspace. In case of the heat equation with Dirichlet boundary conditions the operator grad_0 is one-to-one and so the projection onto the nullspace of A in this case only acts in the second component and is thus easier to handle.

To illustrate the versatility and applicability of Theorem 4.5 we show how our methods apply to the equations of thermopiezoelectricity.

Thermopiezoelectricity

We assume $\Omega \subseteq \mathbb{R}^3$ to be open and bounded. The equations of thermopiezoelectricity describe the interconnected effects of elasticity, thermodynamics and electro-magnetism. The set Ω models a body in its non deformed state. We recall the formulation as in [22, 6.3.3, p. 457], where the model given in [13] is discussed. The unknowns of the system are the time-derivative of the displacement field v , the stress tensor T , the electric and magnetic field E and H as well as the temperature distributions θ and the heat flux Q .

Recalling the spatial derivative operators from the introduction and defining Div and Grad_0 as the divergence on matrix-valued functions and the symmetrized gradient with Dirichlet boundary conditions, respectively, we describe the equations as follows

$$\begin{pmatrix} \varrho_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}d & 0 & C^{-1}\lambda & 0 \\ 0 & d^*C^{-1} & \varepsilon + d^*C^{-1}d & 0 & p + d^*C^{-1}\lambda & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & \lambda^*C^{-1} & p^* + \lambda^*C^{-1}d & 0 & \alpha + \lambda^*C^{-1}\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & q_0 + q_1(\alpha + \kappa\partial_0)^{-1} \end{pmatrix} + \begin{pmatrix} 0 & \text{Div} & 0 & 0 & 0 & 0 \\ \text{Grad}_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{curl} & 0 & 0 \\ 0 & 0 & -\text{curl}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{div} \\ 0 & 0 & 0 & 0 & \text{grad}_0 & 0 \end{pmatrix} \begin{pmatrix} v \\ T \\ E \\ H \\ \theta \\ Q \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ J \\ 0 \\ g \\ 0 \end{pmatrix},$$

where $\varrho_0, C, \varepsilon, \mu, q_0, q_1, \alpha, \kappa, d, \lambda, p$ are bounded linear operators in appropriate $L_2(\Omega)$ -spaces. To frame the latter system into the general context of this exposition, we find that

$$A = \begin{pmatrix} 0 & \text{Div} & 0 & 0 & 0 & 0 \\ \text{Grad}_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{curl} & 0 & 0 \\ 0 & 0 & -\text{curl}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \text{div} \\ 0 & 0 & 0 & 0 & \text{grad}_0 & 0 \end{pmatrix}.$$

The material law is given by

$$M(\partial_0^{-1}) = \begin{pmatrix} \varrho_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}d & 0 & C^{-1}\lambda & 0 \\ 0 & d^*C^{-1} & \varepsilon + d^*C^{-1}d & 0 & p + d^*C^{-1}\lambda & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & \lambda^*C^{-1} & p^* + \lambda^*C^{-1}d & 0 & \alpha + \lambda^*C^{-1}\lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & q_0 + q_1(\alpha + \kappa\partial_0)^{-1} \end{pmatrix},$$

M is a (c) -material law, if we assume that one of the following conditions is satisfied

- (i) $\varrho_0, C, \varepsilon, \mu, q_0, \alpha - p^*\varepsilon^{-1}p, \kappa$ are selfadjoint and strictly positive,
- (ii) $\varrho_0, C, \varepsilon, \mu, q_1, \alpha - p^*\varepsilon^{-1}p, \kappa$ are selfadjoint, strictly positive, $q_1\kappa^{-1} = \kappa^{-1}q_1$ and $q_0 = 0$.

Indeed, the material law can be written as a block operator matrix in the form

$$M(\partial_0^{-1}) = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22}(\partial_0^{-1}) \end{pmatrix},$$

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where

$$\begin{aligned}
 M_{11} &= \begin{pmatrix} \varrho_0 & 0 & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}d & 0 & C^{-1}\lambda \\ 0 & d^*C^{-1} & \varepsilon + d^*C^{-1}d & 0 & p + d^*C^{-1}\lambda \\ 0 & 0 & 0 & \mu & 0 \\ 0 & \lambda^*C^{-1} & p^* + \lambda^*C^{-1}d & 0 & \alpha + \lambda^*C^{-1}\lambda \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & d^* & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \lambda^* & p^*\varepsilon^{-1} & 0 & 1 \end{pmatrix} \begin{pmatrix} \varrho_0 & 0 & 0 & 0 & 0 \\ 0 & C^{-1} & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & \alpha - p^*\varepsilon^{-1}p \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & d & 0 & \lambda \\ 0 & 0 & 1 & 0 & \varepsilon^{-1}p \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}
 \end{aligned}$$

is strictly positive and, by choosing $\nu > 0$ sufficiently large such that $\|\partial_0^{-1}\|$ becomes small enough,

$$M_{22}(\partial_0^{-1}) = q_0 + q_1(\alpha + \kappa\partial_0)^{-1} = q_0 + q_1\kappa^{-1}\partial_0^{-1} + \sum_{n=1}^{\infty} (-1)^n \partial_0^{-n-1} (\kappa^{-1}\alpha)^n \kappa^{-1}$$

is such that either $M_{22}(0)$ or $M'_{22}(0)$ is strictly positive. Thus, M is a (c) -material law for some $c > 0$.

Considering a sequence $(M_n)_n$ of such material laws with the boundedness and uniform positive definiteness assumptions from Corollary 4.7, one may derive a homogenization result for these equations. We will not do this explicitly here. However, in order to satisfy the range condition, one has to assume that all entries of the material law sequence satisfy either condition (i) or (ii). To deduce that A has the (NC) -property, we have to impose suitable geometric requirements on Ω as in the previous example.

We refer the interested reader to more examples of first order formulations of standard evolutionary equations in mathematical physics to [21, 22]. With these formulations it is then straightforward to see when and how our homogenization result applies.

6 Auxiliary results on 0-analytic material laws

In this section, we provide the remaining theorems needed in Section 4. Our main concern will be the discussion of 0-analytic material laws, i.e., material laws that are analytic at $0 \in \mathbb{C}$, cf. [21, 3.3 Some special cases]. To establish Theorem 4.5 similarity transformations of 0-analytic material laws have to be discussed, where our main interest is to show that under any of these similarity transformations a (c) -material law transforms into a (c') -material law for suitable $c' > 0$. In order to achieve the main goal of this section, Theorem 6.11, some technical results are required. We start with a fact concerning Hardy space functions.

Lemma 6.1. *Let X be a Banach space, $\varepsilon > 0$, $\mu(\cdot) = \sum_{n=0}^{\infty} (\cdot)^n \mu_n \in \mathcal{H}^{\infty}(B(0, \varepsilon); X)$. Then for all $k, n \in \mathbb{N}_0$ we have*

$$(i) \quad \|\mu_n\| \leq \|\mu\|_{\infty} \left(\frac{2}{\varepsilon}\right)^n$$

$$(ii) \quad \left\| \sum_{n=k}^{\infty} z^{n-k} \mu_n \right\| \leq 2 \|\mu\|_{\infty} \left(\frac{2}{\varepsilon}\right)^k \text{ for all } z \in B\left(0, \frac{\varepsilon}{4}\right).$$

Proof. The first assertion follows immediately from Cauchy's integral formula (integrate over a circle around 0 with radius $\varepsilon/2$) and the second is a straightforward consequence of the first. \square

With these estimates, we can establish some structural properties of 0-analytic material laws. Recall that the inner products discussed here are linear in the second and conjugate linear in the first component.

Proposition 6.2. *Let H be a Hilbert space, $\varepsilon, c > 0$, $0 < r < \varepsilon/2$. Let M be a material law in $\mathcal{H}^{\infty}(B(0, \varepsilon); L(H)) \cap \mathcal{H}^{\infty, c}(B(r, r); L(H))$. Then $M(0)$ is selfadjoint. For $\phi = \phi_1 \oplus \phi_2 \in R(M(0)) \oplus N(M(0))$, the inequalities*

$$\langle M(0)\phi_1, \phi_1 \rangle \geq 0 \text{ and } \langle \operatorname{Re} M'(0)\phi_2, \phi_2 \rangle \geq c \langle \phi_2, \phi_2 \rangle$$

hold. If, in addition, $R(M(0)) \subseteq H$ is closed, there exists $d > 0$, such that for $\phi_1 \in R(M(0))$ we have

$$\langle M(0)\phi_1, \phi_1 \rangle \geq d \langle \phi_1, \phi_1 \rangle.$$

Proof. We expand M into a power series about 0: $M(z) = \sum_{n=0}^{\infty} z^n M_n$ for $z \in B(0, \varepsilon)$ and suitable $(M_n)_n$ in $L(H)$. Then $M(0) = M_0$ and $M'(0) = M_1$. For $\phi \in H$ define $x_{\phi} := \operatorname{Im} \langle M(0)\phi, \phi \rangle$ and $y_{\phi} := \operatorname{Re} \langle M(0)\phi, \phi \rangle$. It is easy to see that $T : B(r, r) \rightarrow \mathbb{R}_{> \frac{1}{2r}} + i\mathbb{R} : z \mapsto z^{-1}$ is homeomorphic. Thus, for $z \in B(r, r)$ with $z_1 := \operatorname{Im} T(z)$, $z_2 := \operatorname{Re} T(z)$, we have

$$\begin{aligned} c \langle \phi, \phi \rangle &\leq \operatorname{Re} \langle \phi, z^{-1} M(z) \phi \rangle \\ &= \operatorname{Re}(iz_1 + z_2)(ix_{\phi} + y_{\phi}) + \operatorname{Re} \langle \phi, \sum_{n=1}^{\infty} z^{n-1} M_n \phi \rangle \\ &= -z_1 x_{\phi} + z_2 y_{\phi} + \operatorname{Re} \langle \phi, \sum_{n=1}^{\infty} z^{n-1} M_n \phi \rangle. \end{aligned}$$

The left-hand side is non-negative. The last term on the right-hand side is bounded for $z \rightarrow 0$. Moreover, since T is bijective (in particular, for every z_2 the values of z_1 range over the whole real axis), it follows that $x_{\phi} = 0$. Thus, we arrive at

$$c \langle \phi, \phi \rangle \leq z_2 y_{\phi} + \operatorname{Re} \langle \phi, \sum_{n=1}^{\infty} z^{n-1} M_n \phi \rangle.$$

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Now, since z_2 can be chosen arbitrarily large, while the second term of the right-hand side remains bounded, it follows that $y_\phi \geq 0$. Thus, for every $\phi \in H$, we deduce that $\langle \phi, M(0)\phi \rangle \geq 0$. Since $M(0)$ is a bounded operator in the complex Hilbert space H , the operator $M(0)$ is selfadjoint and positive (semi-)definite and therefore $H = \overline{R(M(0))} \oplus N(M(0))$. Let $\phi \in N(M(0))$. Then for $\varepsilon/2 > \eta > 0$,

$$\begin{aligned} c|\phi|^2 &\leq \operatorname{Re} \langle \phi, \sum_{k=1}^{\infty} \eta^{k-1} M_k \phi \rangle \\ &= \operatorname{Re} \langle \phi, M'(0)\phi \rangle + \eta \operatorname{Re} \langle \phi, \sum_{k=2}^{\infty} \eta^{k-2} M_k \phi \rangle. \end{aligned}$$

If we let $\eta \rightarrow 0+$, we get that $\operatorname{Re} \langle \phi, M'(0)\phi \rangle \geq c|\phi|^2$. Now, $M(0)$ is invariant on its range and the restriction of $M(0)$ to its range is one-to-one. Thus, if $R(M(0))$ is closed, the closed graph theorem implies that $M(0) : R(M(0)) \rightarrow R(M(0))$ is continuously invertible. By the spectral theorem for continuous and selfadjoint operators it follows that $M(0)$ is strictly positive on its range. \square

Remark 6.3. We shall note here that the converse of Proposition 6.2 is also true in the following sense: Let $M \in \mathcal{H}^\infty(B(0, \varepsilon); L(H))$ be such that $M(0) = M(0)^*$. Assume that there exist $d, c > 0$ such that for all $\phi_1 \in R(M(0)), \phi_2 \in N(M(0))$

$$\langle M(0)\phi_1, \phi_1 \rangle \geq d \langle \phi_1, \phi_1 \rangle \text{ and } \langle \operatorname{Re} M'(0)\phi_2, \phi_2 \rangle \geq c \langle \phi_2, \phi_2 \rangle.$$

Then $R(M(0)) \subseteq H$ is closed and for $0 < r \leq \frac{1}{2 \max\{\nu_1, \hat{\delta}^{-1}\}}$, $M \in \mathcal{H}^{\infty, c/3}(B(r, r); L(H))$, cf. [27, Lemma 2.3] or [22, Remark 6.2.7], where $\nu_1 := \frac{1}{d} \left(\frac{2c}{3} + \frac{3}{c} \left(\|M\|_\infty \frac{2}{\varepsilon} \right)^2 + \frac{2}{\varepsilon} \|M\|_\infty \right)$ and $\hat{\delta} := \min\{\|M\|_\infty^{-1} \left(\frac{\varepsilon}{2} \right)^2 \frac{c}{6}, \frac{\varepsilon}{4}\}$.

Proof. It is easy to see that $R(M(0))$ is closed. Let $(M_n)_n$ in $L(H)$ be such that $M(z) = \sum_{n=0}^{\infty} z^n M_n$ for all $z \in B(0, \varepsilon)$. By Lemma 6.1, we have $\|M'(0)\| \leq \frac{2}{\varepsilon} \|M\|_\infty$ and, for all $0 < \delta \leq \varepsilon/4$ and $z \in B(0, \delta)$, $\|\sum_{n=2}^{\infty} z^{n-1} M_n\| \leq 2\delta \left(\frac{2}{\varepsilon} \right)^2 \|M\|_\infty$. For $\nu \geq \max\{\nu_1, \hat{\delta}^{-1}\}$, $z \in B(1/(2\nu), 1/(2\nu))$, $\phi = (\phi_1, \phi_2) \in R(M(0)) \oplus N(M(0))$ and $\eta > 0$,

$$\begin{aligned} &\langle \phi, \operatorname{Re} z^{-1} M(z) \phi \rangle \\ &= (\operatorname{Re} z^{-1}) \langle \phi_1, M(0)\phi_1 \rangle + \langle \phi, \operatorname{Re} M'(0)\phi \rangle + \operatorname{Re} \langle \phi, \sum_{n=2}^{\infty} z^{n-1} M_n \phi \rangle \\ &\geq \nu d |\phi_1|^2 + c |\phi_2|^2 - 2 \|M'(0)\| |\phi_1| |\phi_2| - \|M'(0)\| |\phi_1|^2 - 2\hat{\delta} \|M\|_\infty \left(\frac{2}{\varepsilon} \right)^2 |\phi|^2 \\ &\geq \left(\nu d - \eta \|M'(0)\|^2 - \|M'(0)\| \right) |\phi_1|^2 + \left(c - \frac{1}{\eta} \right) |\phi_2|^2 - \frac{c}{3} |\phi|^2 \end{aligned}$$

$$\geq \left(\nu d - \eta \left(\|M\|_\infty \frac{2}{\varepsilon} \right)^2 - \|M\|_\infty \frac{2}{\varepsilon} - \frac{c}{3} \right) |\phi_1|^2 + \left(\frac{2c}{3} - \frac{1}{\eta} \right) |\phi_2|^2.$$

If $\eta = 3/c$, using $\nu > \nu_1$, we obtain

$$\langle \operatorname{Re} z^{-1} M(z) \phi, \phi \rangle \geq \left(\nu d - \frac{3}{c} \left(\|M\|_\infty \frac{2}{\varepsilon} \right)^2 - \|M\|_\infty \frac{2}{\varepsilon} - \frac{c}{3} \right) |\phi_1|^2 + \frac{c}{3} |\phi_2|^2 \geq \frac{c}{3} |\phi|^2. \quad \square$$

This completes the general discussion on 0-analytic material laws. With the main results of Section 4 in mind, it is natural to discuss the following situation for material laws M :

Assumption 6.4. Assume there exist Hilbert spaces H_1, H_2 and constants $\varepsilon, c > 0$, $0 < r < \varepsilon/2$ with

$$M \in \mathcal{H}^\infty(B(0, \varepsilon); L(H_1 \oplus H_2)) \cap \mathcal{H}^{\infty, c}(B(r, r); L(H_1 \oplus H_2))$$

and $R(M(0)) \subseteq H_1 \oplus H_2$ is closed.

Before we turn to Gauss-transformations on the material law, we study some structural properties of a material law satisfying Assumption 6.4. These properties are stated in the next theorem for which we need the following elementary prerequisite.

Lemma 6.5. *Let H_1, H_2 be Hilbert spaces. Assume that*

$$\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in L(H_1 \oplus H_2)$$

is selfadjoint and positive definite. Then $M_{12} = M_{21}^$ and if $M_{22} = 0$ then $M_{12} = 0$.*

Proof. It is easy to see that $M_{11} = M_{11}^*$ and $M_{22} = M_{22}^*$ and thus

$$\begin{pmatrix} 0 & M_{12} \\ M_{21} & 0 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} - \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix}$$

is selfadjoint. Assume now that $M_{22} = 0$. If $M_{12} \neq 0$, then there exists $(\phi_1, \phi_2) \in H_1 \oplus H_2$ such that $\operatorname{Re} \langle M_{12} \phi_2, \phi_1 \rangle = \langle M_{12} \phi_2, \phi_1 \rangle + \langle M_{21} \phi_1, \phi_2 \rangle < 0$. For $\alpha > 0$ we deduce that

$$0 \leq \left\langle \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \alpha \phi_2 \end{pmatrix}, \begin{pmatrix} \phi_1 \\ \alpha \phi_2 \end{pmatrix} \right\rangle = \langle M_{11} \phi_1, \phi_1 \rangle + \alpha (\langle M_{12} \phi_2, \phi_1 \rangle + \langle M_{21} \phi_1, \phi_2 \rangle),$$

which yields a contradiction if α is chosen large enough. \square

In the following, for Hilbert spaces H_1, H_2 we denote the canonical orthogonal projection $H_1 \oplus H_2 \rightarrow H_j$ onto the j th coordinate by π_j , $j \in \{1, 2\}$.

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Theorem 6.6. *Let M satisfy Assumption 6.4. Using the notation from Assumption 6.4, we define*

$$G_1 := R(\pi_1 M(0) \pi_1^*), \quad G_2 := N(\pi_1 M(0) \pi_1^*), \quad G_3 := R(\pi_2 M(0) \pi_2^*), \quad G_4 := N(\pi_2 M(0) \pi_2^*).$$

Then M has the following form:

$$M = \left(z \mapsto \begin{pmatrix} M_{11}^{(0)} & 0 & M_{13}^{(0)} & 0 \\ 0 & 0 & 0 & 0 \\ M_{31}^{(0)} & 0 & M_{33}^{(0)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + z \begin{pmatrix} M_{11}^{(1)}(z) & M_{12}^{(1)}(z) & M_{13}^{(1)}(z) & M_{14}^{(1)}(z) \\ M_{21}^{(1)}(z) & M_{22}^{(1)}(z) & M_{23}^{(1)}(z) & M_{24}^{(1)}(z) \\ M_{31}^{(1)}(z) & M_{32}^{(1)}(z) & M_{33}^{(1)}(z) & M_{34}^{(1)}(z) \\ M_{41}^{(1)}(z) & M_{42}^{(1)}(z) & M_{43}^{(1)}(z) & M_{44}^{(1)}(z) \end{pmatrix} \right) \in \mathcal{H}^\infty \left(B(0, \varepsilon); L \left(\bigoplus_{j=1}^4 G_j \right) \right),$$

where for $j, k \in \{1, 2, 3, 4\}$ we have $M_{jk}^{(1)} \in \mathcal{H}^\infty(B(0, \varepsilon); L(G_k, G_j))$ and if $j, k \in \{1, 3\}$ we have $M_{kj}^{(0)*} = M_{jk}^{(0)} \in L(G_k, G_j)$. Moreover, there is $d > 0$ such that $M_{jj}^{(0)} \geq d$ for $j \in \{1, 3\}$.

Proof. By Proposition 6.2, we know that $M(0)$ is selfadjoint and strictly positive definite on its range. Thus,

$$M_{jj}^{(0)} = (\pi_j^* M(0) \pi_j : G_j \rightarrow G_j)$$

is selfadjoint and strictly positive definite and therefore $H_{\frac{1}{2}j + \frac{1}{2}} = G_j \oplus G_{j+1}$ for $j \in \{1, 3\}$.

We denote by $\varrho_j : H_1 \oplus H_2 \rightarrow G_j$ the orthogonal projections onto G_j and define $M_{jk}^{(0)} := \varrho_j M(0) \varrho_k^*$, $M_{jk}^{(1)} := \varrho_j (M - M(0)) \varrho_k^*$ for all $j, k \in \{1, 2, 3, 4\}$. Hence,

$$M = \left(z \mapsto \begin{pmatrix} M_{11}^{(0)} & 0 & M_{13}^{(0)} & M_{14}^{(0)} \\ 0 & 0 & M_{23}^{(0)} & M_{24}^{(0)} \\ M_{31}^{(0)} & M_{32}^{(0)} & M_{33}^{(0)} & 0 \\ M_{41}^{(0)} & M_{42}^{(0)} & 0 & 0 \end{pmatrix} + z \begin{pmatrix} M_{11}^{(1)}(z) & M_{12}^{(1)}(z) & M_{13}^{(1)}(z) & M_{14}^{(1)}(z) \\ M_{21}^{(1)}(z) & M_{22}^{(1)}(z) & M_{23}^{(1)}(z) & M_{24}^{(1)}(z) \\ M_{31}^{(1)}(z) & M_{32}^{(1)}(z) & M_{33}^{(1)}(z) & M_{34}^{(1)}(z) \\ M_{41}^{(1)}(z) & M_{42}^{(1)}(z) & M_{43}^{(1)}(z) & M_{44}^{(1)}(z) \end{pmatrix} \right).$$

As $M(0)$ is selfadjoint, Lemma 6.5 shows that $M_{kj}^{(0)*} = M_{jk}^{(0)} \in L(G_k, G_j)$ for all $k, j \in \{1, 2, 3, 4\}$. Since $M(0)$ is positive definite, it follows that the block operator matrices

$$\begin{pmatrix} M_{11}^{(0)} & 0 & 0 & M_{14}^{(0)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ M_{41}^{(0)} & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & M_{23}^{(0)} & 0 \\ 0 & M_{32}^{(0)} & M_{33}^{(0)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_{24}^{(0)} \\ 0 & 0 & 0 & 0 \\ 0 & M_{42}^{(0)} & 0 & 0 \end{pmatrix}$$

are positive definite as well. Thus, by Lemma 6.5, we deduce that $M_{14}^{(0)} = M_{41}^{(0)*} = 0$, $M_{23}^{(0)} = M_{32}^{(0)*} = 0$ and $M_{24}^{(0)} = M_{42}^{(0)*} = 0$. \square

For the next theorem we note that, for Hilbert spaces H_1, H_2 and $B \in L(H_2, H_1)$, we have

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix} \text{ and } \left\| \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}^{-1} \right\| \leq \sqrt{1 + \|B\| + \|B\|^2}. \quad (6.1)$$

Moreover, we need the following lemmas:

Lemma 6.7. *Let H be a Hilbert space. Let $T \in L(H)$ be continuously invertible, $A, B \in L(H)$. If $A = T^*BT$, i.e., A and B are similar, then $\operatorname{Re} A = T^* \operatorname{Re} BT$ and if in addition $\operatorname{Re} B \geq c > 0$ then $\operatorname{Re} A \geq \frac{c}{\|T^{-1}\|^2}$.*

Proof. We have $2 \operatorname{Re} A = A^* + A = T^*B^*T + T^*BT = 2T^* \operatorname{Re} BT$. Assume that $\operatorname{Re} B \geq c$ for some $c > 0$. Then, for $\phi \in H$,

$$\langle \operatorname{Re} A \phi, \phi \rangle = \langle \operatorname{Re} BT \phi, T \phi \rangle \geq c \langle T \phi, T \phi \rangle \geq \frac{c}{\|T^{-1}\|^2} \langle \phi, \phi \rangle. \quad \square$$

Lemma 6.8. *Let M satisfy Assumption 6.4. Using the notation from Assumption 6.4, we define*

$$G_1 := R(\pi_1 M(0) \pi_1^*), \quad G_2 := N(\pi_1 M(0) \pi_1^*), \quad G_3 := R(\pi_2 M(0) \pi_2^*), \quad G_4 := N(\pi_2 M(0) \pi_2^*).$$

Let $N_{13}^{(0)} \in L(G_3, G_1)$, $N_{14}^{(0)} \in L(G_4, G_1)$, $N_{41}^{(0)} \in L(G_1, G_4)$, $N_{24}^{(0)} \in L(G_4, G_2)$, $N_1^{(1)} \in \mathcal{H}^\infty(B(0, \varepsilon); L(G_3 \oplus G_4, G_1 \oplus G_2))$, $N_{1'}^{(1)} \in \mathcal{H}^\infty(B(0, \varepsilon); L(G_1 \oplus G_2, G_3 \oplus G_4))$ and

$$\begin{aligned} N_1 &:= \left(z \mapsto \begin{pmatrix} N_{13}^{(0)} & N_{14}^{(0)} \\ 0 & N_{24}^{(0)} \end{pmatrix} + z N_1^{(1)}(z) \right) \\ N_{1'} &:= \left(z \mapsto \begin{pmatrix} N_{13}^{(0)*} & 0 \\ N_{41}^{(0)} & N_{24}^{(0)*} \end{pmatrix} + z N_{1'}^{(1)}(z) \right). \end{aligned}$$

Then,

$$\mathcal{M} := \left(z \mapsto \begin{pmatrix} 1 & N_1(z) \\ 0 & 1 \end{pmatrix} M(z) \begin{pmatrix} 1 & 0 \\ N_{1'}(z) & 1 \end{pmatrix} \right) \in \mathcal{H}^\infty(B(0, \varepsilon); L(H_1 \oplus H_2))$$

and $R(M(0)) = R(\mathcal{M}(0))$, $\mathcal{M}(0) \geq d'$ on its range and $\operatorname{Re} \mathcal{M}'(0) \geq c'$ on the nullspace of $\mathcal{M}(0)$, where

$$d' := d \left(\sqrt{1 + \|N_{13}^{(0)}\| + \|N_{13}^{(0)}\|^2} \right)^{-2} \quad \text{and} \quad c' := c \left(\sqrt{1 + \|N_{24}^{(0)}\| + \|N_{24}^{(0)}\|^2} \right)^{-2},$$

with $d > 0$ being the constant of positive definiteness of $M(0)$ on its range from Theorem 6.6.

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Proof. Using the representation of M given in Theorem 6.6, we compute $\mathcal{M}(0)$:

$$\begin{aligned}\mathcal{M}(0) &= \begin{pmatrix} 1 & \begin{pmatrix} N_{13}^{(0)} & N_{14}^{(0)} \\ 0 & N_{24}^{(0)} \end{pmatrix} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_{11}^{(0)} & 0 & M_{13}^{(0)} & 0 \\ 0 & 0 & 0 & 0 \\ M_{31}^{(0)} & 0 & M_{33}^{(0)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \begin{pmatrix} N_{13}^{(0)*} & 0 \\ N_{41}^{(0)} & N_{42}^{(0)*} \end{pmatrix} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & \begin{pmatrix} N_{13}^{(0)} & 0 \\ 0 & 0 \end{pmatrix} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} M_{11}^{(0)} & 0 & M_{13}^{(0)} & 0 \\ 0 & 0 & 0 & 0 \\ M_{31}^{(0)} & 0 & M_{33}^{(0)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \begin{pmatrix} N_{13}^{(0)*} & 0 \\ 0 & 0 \end{pmatrix} & 1 \end{pmatrix}.\end{aligned}$$

Hence, $\mathcal{M}(0)$ is similar to a positive definite operator. Moreover, the similarity transformation commutes with the projector

$$P := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, $M(0)$ and $\mathcal{M}(0)$ have the same range and are both strictly positive on it. Indeed, $M(0)$ is strictly positive on $G_1 \oplus \{0\} \oplus G_3 \oplus \{0\}$ and since the similarity transformation is a bijection on $G_1 \oplus \{0\} \oplus G_3 \oplus \{0\}$, $\mathcal{M}(0)$ is a bijection on $G_1 \oplus \{0\} \oplus G_3 \oplus \{0\}$ as well. In view of Lemma 6.7 and Inequality (6.1), we deduce $\mathcal{M}(0) \geq d'$ on its range. Next, consider $(1 - P)\mathcal{M}'(0)(1 - P)$. For this purpose, we compute

$$\begin{aligned}&(1 - P) \begin{pmatrix} 1 & N_1(z) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} N_{13}^{(0)} & N_{14}^{(0)} \\ 0 & N_{24}^{(0)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} z N_1^{(1)}(z) \right) \\ &= \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & N_{24}^{(0)} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} z \begin{pmatrix} 0 & 0 \\ N_{1,23}^{(1)}(z) & N_{1,24}^{(1)}(z) \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}\end{aligned}$$

with suitable $N_{1,2k}^{(1)} \in \mathcal{H}^\infty(B(0, \varepsilon); L(G_k, G_2))$ ($k \in \{3, 4\}$) and, similarly, we find

$$\begin{pmatrix} 1 & 0 \\ N_{1'}(z) & 1 \end{pmatrix} (1 - P) = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & N_{24}^{(0)*} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} + \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ z \begin{pmatrix} 0 & N_{1',32}^{(1)}(z) \\ 0 & N_{1',42}^{(1)}(z) \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

for suitable $N_{1',k2}^{(1)} \in \mathcal{H}^\infty(B(0, \varepsilon); L(G_2, G_k))$ ($k \in \{3, 4\}$). Note that $\mathcal{M}'(0)$ consists only of the first order terms of \mathcal{M} . Since the zeroth order term of M leaves the range of P invariant and, due to the structure of the transformations $\begin{pmatrix} 1 & 0 \\ N_{1'}(z) & 1 \end{pmatrix} (1 - P)$ and $(1 - P) \begin{pmatrix} 1 & N_1(z) \\ 0 & 1 \end{pmatrix}$, most of the terms cancel, we arrive at

$$\begin{aligned} & (1 - P)\mathcal{M}'(0)(1 - P) \\ &= \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & N_{24}^{(0)} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} M_{11}^{(1)}(0) & M_{12}^{(1)}(0) & M_{13}^{(1)}(0) & M_{14}^{(1)}(0) \\ M_{21}^{(1)}(0) & M_{22}^{(1)}(0) & M_{23}^{(1)}(0) & M_{24}^{(1)}(0) \\ M_{31}^{(1)}(0) & M_{32}^{(1)}(0) & M_{33}^{(1)}(0) & M_{34}^{(1)}(0) \\ M_{41}^{(1)}(0) & M_{42}^{(1)}(0) & M_{43}^{(1)}(0) & M_{44}^{(1)}(0) \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & N_{24}^{(0)*} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & N_{24}^{(0)} \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & M_{22}^{(1)}(0) & 0 & M_{24}^{(1)}(0) \\ 0 & 0 & 0 & 0 \\ 0 & M_{42}^{(1)}(0) & 0 & M_{44}^{(1)}(0) \end{pmatrix} \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & N_{24}^{(0)*} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \end{aligned}$$

Now, by Lemma 6.7 and Inequality (6.1), we see $\operatorname{Re} \mathcal{M}'(0) \geq c'$ on the nullspace of $\mathcal{M}(0)$. \square

Remark 6.9. Consider the following situation where we apply Lemma 6.8. Let G_j be a Hilbert space for $j \in \{1, 2, 3, 4\}$. Let

$$\begin{aligned} N_2 &:= \left(z \mapsto \begin{pmatrix} N_{13}^{(0)} & 0 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} N_{13}^{(1)}(z) & N_{14}^{(1)}(z) \\ N_{23}^{(1)}(z) & N_{24}^{(1)}(z) \end{pmatrix} \right) \in \mathcal{H}^\infty(B(0, \varepsilon); L(G_3 \oplus G_4, G_1 \oplus G_2)) \\ N_{2'} &:= \left(z \mapsto \begin{pmatrix} N_{31}^{(0)} & 0 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} N_{31}^{(1)}(z) & N_{32}^{(1)}(z) \\ N_{41}^{(1)}(z) & N_{42}^{(1)}(z) \end{pmatrix} \right) \in \mathcal{H}^\infty(B(0, \varepsilon); L(G_1 \oplus G_2, G_3 \oplus G_4)) \end{aligned}$$

and assume $N_{13}^{(0)*} = N_{31}^{(0)} \in L(G_3, G_1)$. Moreover, let

$$N_3 := \left(z \mapsto \begin{pmatrix} N_{33}^{(0)} + zN_{33}^{(1)}(z) & N_{34}^{(1)}(z) \\ N_{43}^{(1)}(z) & z^{-1}N_{44}^{(0)} + N_{44}^{(1)}(z) \end{pmatrix} \right)$$

with $N_{33}^{(0)*} = N_{33}^{(0)} \in L(G_3)$, $N_{44}^{(0)} \in L(G_4)$, $N_{jk}^{(1)} \in \mathcal{H}^\infty(B(0, \varepsilon); L(G_k, G_j))$, $j, k \in \{3, 4\}$. Then it is easy to see that $N_2(\cdot)N_3(\cdot) \in \mathcal{H}^\infty(B(0, \varepsilon); L(G_3 \oplus G_4, G_1 \oplus G_2))$ and $N_3(\cdot)N_{2'}(\cdot) \in$

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$\mathcal{H}^\infty(B(0, \varepsilon); L(G_1 \oplus G_2, G_3 \oplus G_4))$ and the following expansions hold

$$N_2(z)N_3(z) = \begin{pmatrix} N_{13}^{(0)}N_{33}^{(0)} & N_{13}^{(0)}N_{34}^{(1)}(0) \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & N_{14}^{(1)}(0)N_{44}^{(0)} \\ 0 & N_{24}^{(1)}(0)N_{44}^{(0)} \end{pmatrix} + O(z)$$

and

$$N_3(z)N_{2'}(z) = \begin{pmatrix} N_{33}^{(0)}N_{31}^{(0)} & 0 \\ N_{43}^{(1)}(0)N_{31}^{(0)} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ N_{44}^{(0)}N_{41}^{(1)}(0) & N_{44}^{(0)}N_{42}^{(1)}(0) \end{pmatrix} + O(z)$$

for $z \rightarrow 0$. Now, let M and the G_j 's be as in Lemma 6.8. Assume the following compatibility condition

$$\left(N_{44}^{(0)}N_{42}^{(1)}(0)\right)^* = N_{24}^{(1)}(0)N_{44}^{(0)}.$$

Then $N_1 := N_2N_3$ and $N_{1'} := N_3N_{2'}$ satisfy the assumptions from Lemma 6.8.

We now turn to the analysis inverses of material laws. Since we need to estimate the norm bounds of these inverses, we observe that, for a Hilbert space H and a continuous linear operator $B \in L(H)$ satisfying $\operatorname{Re} B \geq h$ for some $h > 0$, $B^{-1} \in L(H)$. Moreover, using the Cauchy-Schwarz inequality, we deduce the estimate

$$\|B^{-1}\| \leq 1/h. \quad (6.2)$$

Another consequence of $\operatorname{Re} B \geq h$ is

$$\operatorname{Re} B^{-1} \geq h/(\|B\|^2). \quad (6.3)$$

Further, note that, for appropriate linear operators $\alpha, \beta, \gamma, \delta$ in Hilbert spaces and $A := (\alpha - \beta\delta^{-1}\gamma)^{-1}$, we have formally

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} A & -A\beta\delta^{-1} \\ -\delta^{-1}\gamma A & \delta^{-1}\gamma A\beta\delta^{-1} + \delta^{-1} \end{pmatrix}.$$

Lemma 6.10. *Let H_1, H_2 be Hilbert spaces, $d, c, \varepsilon > 0$. Let $L(H_1) \ni M_{11}^{(0)} = M_{11}^{(0)*} \geq d$. Moreover, let $M_{jk}^{(1)} \in \mathcal{H}^\infty(B(0, \varepsilon); L(H_k, H_j))$ with $\operatorname{Re} M_{22}^{(1)}(0) \geq c$. Define*

$$M_I := \left(B(0, \varepsilon) \ni z \mapsto \begin{pmatrix} M_{11}^{(0)} & 0 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} M_{11}^{(1)}(z) & M_{12}^{(1)}(z) \\ M_{21}^{(1)}(z) & M_{22}^{(1)}(z) \end{pmatrix} \right).$$

Then there exists $\varepsilon' > 0$ depending on $\varepsilon, \|M_I\|_\infty, c$ and d such that, for $z \in B(0, \varepsilon') \setminus \{0\}$,

$$\begin{aligned} & M_I(z)^{-1} \\ &= \begin{pmatrix} M_{121}(z) & -M_{121}(z)M_{12}^{(1)}(z)M_{22}^{(1)}(z)^{-1} \\ -M_{22}^{(1)}(z)^{-1}M_{21}^{(1)}(z)M_{121}(z) & M_{22}^{(1)}(z)^{-1}M_{21}^{(1)}(z)M_{121}(z)M_{12}^{(1)}(z)M_{22}^{(1)}(z)^{-1} + z^{-1}M_{22}^{(1)}(z)^{-1} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} M_{121}(z) &:= \left(M_{11}^{(0)} + z \left(M_{11}^{(1)}(z) - M_{12}^{(1)}(z) M_{22}^{(1)}(z)^{-1} M_{21}^{(1)}(z) \right) \right)^{-1} \\ &= \left(M_{11}^{(0)} \right)^{-1} + O(z) \text{ and} \\ M_{22}^{(1)}(z)^{-1} &= M_{22}^{(1)}(0)^{-1} + O(z). \end{aligned}$$

On the other hand, for $\widehat{M}_{jk} \in \mathcal{H}^\infty(B(0, \varepsilon'); L(H_k, H_j))$ with $\operatorname{Re} \widehat{M}_{22}^{(1)}(0) \geq c$ and

$$\widehat{M}_I := \left(B(0, \varepsilon') \setminus \{0\} \ni z \mapsto \begin{pmatrix} M_{11}^{(0)} + z \widehat{M}_{11}^{(1)}(z) & \widehat{M}_{12}^{(1)}(z) \\ \widehat{M}_{21}^{(1)}(z) & z^{-1} \widehat{M}_{22}^{(1)}(z) \end{pmatrix} \right)$$

there exists $\varepsilon'' > 0$ depending on $c, d, \left\| \widehat{M}_{jk} \right\|_\infty$ ($j, k \in \{1, 2\}$) such that, for all $z \in B(0, \varepsilon'')$,

$$\begin{aligned} \widehat{M}_I(z)^{-1} &= \begin{pmatrix} \widehat{M}_{121}(z) & -z \widehat{M}_{121}(z) \widehat{M}_{12}^{(1)}(z) \widehat{M}_{22}^{(1)}(z)^{-1} \\ -z \widehat{M}_{22}^{(1)}(z)^{-1} \widehat{M}_{21}^{(1)}(z) \widehat{M}_{121}(z) & z^2 \widehat{M}_{22}^{(1)}(z)^{-1} \widehat{M}_{21}^{(1)}(z) \widehat{M}_{121}(z) M_{12}^{(1)}(z) \widehat{M}_{22}^{(1)}(z)^{-1} + z \widehat{M}_{22}^{(1)}(z)^{-1} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \widehat{M}_{121}(z) &:= \left(M_{11}^{(0)} + z \left(\widehat{M}_{11}^{(1)}(z) - \widehat{M}_{12}^{(1)}(z) z \widehat{M}_{22}^{(1)}(z)^{-1} \widehat{M}_{21}^{(1)}(z) \right) \right)^{-1} \\ &= \left(M_{11}^{(0)} \right)^{-1} + O(z) \text{ and} \\ \widehat{M}_{22}^{(1)}(z)^{-1} &= \widehat{M}_{22}^{(1)}(0)^{-1} + O(z). \end{aligned}$$

In particular, $M_{I'} \in \mathcal{H}^\infty(B(0, \varepsilon''); L(H_1 \oplus H_2))$.

Proof. The expressions for the inverses of M_I and $M_{I'}$ can be verified immediately. The asymptotic expansions are straightforward applications of the Neumann series expansion. The respective convergence radii can be estimated in terms of $\varepsilon, \|M_I\|_\infty, c$ and d or $\varepsilon, \left\| \widehat{M}_{jk} \right\|_\infty$ ($j, k \in \{1, 2\}$), c and d by Lemma 6.1 and Inequality (6.2). \square

Theorem 6.11. Let H_1, H_2 be separable Hilbert spaces, $c, d, \varepsilon, r > 0$. Let $(N_n)_n = \left(\begin{pmatrix} N_{11,n} & N_{12,n} \\ N_{21,n} & N_{22,n} \end{pmatrix} \right)_n$ be a bounded sequence in $\mathcal{H}^\infty(B(0, \varepsilon); L(H_1 \oplus H_2)) \cap \mathcal{H}^{\infty, c}(B(r, r); L(H_1 \oplus H_2))$ and $M \in \mathcal{H}^\infty(B(0, \varepsilon); L(H_1 \oplus H_2)) \cap \mathcal{H}^{\infty, c}(B(r, r); L(H_1 \oplus H_2))$. Assume that for all $n \in \mathbb{N}$ we have $R(M(0)) = R(N_n(0))$ and $M(0), N_n(0) \geq d$ on $R(M(0))$. Denote by $q_j : H_j \rightarrow R(\pi_j^*) \cap N(M(0))$ ($j \in \{1, 2\}$) the canonical ortho-projections. Assume for all $n \in \mathbb{N}$ the compatibility condition

$$\left((q_2 N'_{22,n}(0) q_2^*)^{-1} q_2 N'_{21,n}(0) q_1^* \right)^* = q_1 N'_{12,n}(0) q_2^* (q_2 N'_{22,n}(0) q_2^*)^{-1}.$$

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Then there exist $\varepsilon', r', c' > 0$ depending on $\varepsilon, c, r, \|M\|_\infty, \sup\{\|N_n\|_\infty; n \in \mathbb{N}\}, d$ such that

$$\begin{aligned} (N_{1,n})_n &:= \left((z \mapsto N_{12,n}(z) N_{22,n}(z)^{-1}) \right)_n \\ (N_{1',n})_n &:= \left((z \mapsto N_{22,n}(z)^{-1} N_{21,n}(z)) \right)_n \end{aligned}$$

are bounded in $\mathcal{H}^\infty(B(0, \varepsilon'); L(H_2, H_1))$ and $\mathcal{H}^\infty(B(0, \varepsilon'); L(H_1, H_2))$, respectively. Moreover, denoting by $M_1 \in \mathcal{H}_w^\infty(B(0, \varepsilon'); L(H_2, H_1))$ and $M_{1'} \in \mathcal{H}_w^\infty(B(0, \varepsilon'); L(H_1, H_2))$ the respective limits of $(N_{1,n_k})_k$ and $(N_{1',n_k})_k$ for a strictly monotone sequence of positive integers $(n_k)_k$, we have

$$\begin{aligned} \mathcal{M} &:= \left(z \mapsto \begin{pmatrix} 1 & \pm M_1(z) \\ 0 & 1 \end{pmatrix} M(z) \begin{pmatrix} 1 & 0 \\ \pm M_{1'}(z) & 1 \end{pmatrix} \right) \\ &\in \mathcal{H}^\infty(B(0, \varepsilon'); L(H_1 \oplus H_2)) \cap \mathcal{H}^{\infty, c'}(B(r', r'); L(H_1 \oplus H_2)), \end{aligned}$$

and $R(\mathcal{M}(0)) = R(M(0))$.

Proof. In the following, we use Hilbert spaces G_j , $j \in \{1, 2, 3, 4\}$ as in Theorem 6.6 and represent N_n for $n \in \mathbb{N}$ using bounded operators $N_{jk,n}^{(0)}$, $(j, k) \in \{(1, 1), (1, 3), (3, 1), (3, 3)\}$, and Hardy space functions $N_{jk,n}^{(1)}$, $j, k \in \{1, 2, 3, 4\}$, as in Theorem 6.6. From Lemma 6.10 we have an explicit expression for $N_{22,n}(z)^{-1}$, namely

$$N_{22,n}(z)^{-1} = \begin{pmatrix} \left(N_{33,n}^{(0)} \right)^{-1} + O(z) & O(1) \\ O(1) & z^{-1} N_{44,n}^{(1)}(0)^{-1} + O(1) \end{pmatrix} \text{ for } z \rightarrow 0.$$

Moreover, we have an estimate for the radius of convergence ε' for the Neumann expansion involved in this expression in terms of $\sup\{\|N_n\|_\infty; n \in \mathbb{N}\}, d, c, \varepsilon$. In particular, $z \mapsto N_{22,n}(z)^{-1}$ satisfies the assumptions on N_3 in Remark 6.9 (note that $\left(\left(N_{33,n}^{(0)} \right)^{-1} \right)^* = \left(N_{33,n}^{(0)} \right)^{-1}$, by Theorem 6.6). Moreover, using Theorem 6.6, we deduce that $N_{12,n}$ and $N_{21,n}$ satisfy the assumptions on N_2 and $N_{2'}$ in Remark 6.9. Indeed, we have

$$\begin{aligned} N_{12,n}(z) &:= \left(z \mapsto \begin{pmatrix} N_{13,n}^{(0)} & 0 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} N_{13,n}^{(1)}(z) & N_{14,n}^{(1)}(z) \\ N_{23,n}^{(1)}(z) & N_{24,n}^{(1)}(z) \end{pmatrix} \right) \\ &\in \mathcal{H}^\infty(B(0, \varepsilon); L(G_3 \oplus G_4, G_1 \oplus G_2)) \\ N_{21,n}(z) &:= \left(z \mapsto \begin{pmatrix} N_{31,n}^{(0)} & 0 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} N_{31,n}^{(1)}(z) & N_{32,n}^{(1)}(z) \\ N_{41,n}^{(1)}(z) & N_{42,n}^{(1)}(z) \end{pmatrix} \right) \\ &\in \mathcal{H}^\infty(B(0, \varepsilon); L(G_1 \oplus G_2, G_3 \oplus G_4)) \end{aligned}$$

with $\left(N_{13,n}^{(0)} \right)^* = N_{31,n}^{(0)}$ by Theorem 6.6. Moreover, Remark 6.9 shows that $N_{1,n}$ and $N_{1',n}$

satisfy the assumptions imposed on N_1 and $N_{1'}$ in Lemma 6.8. More precisely, we have the expansions

$$\begin{aligned} N_{1,n}(z) &= \begin{pmatrix} N_{13,n}^{(0)}(N_{33,n}^{(0)})^{-1} & \hat{N}_{14,n} \\ 0 & N_{24,n}^{(1)}(0)N_{44,n}^{(1)}(0)^{-1} \end{pmatrix} + O(z) \\ N_{1',n}(z) &= \begin{pmatrix} (N_{33,n}^{(0)})^{-1}N_{31,n}^{(0)} & 0 \\ \hat{N}_{41,n} & N_{44,n}^{(1)}(0)^{-1}N_{42,n}^{(1)}(0) \end{pmatrix} + O(z) \end{aligned}$$

for suitable continuous linear operators $\hat{N}_{14,n}, \hat{N}_{41,n}$. We deduce that

$$\left(N_{13,n}^{(0)}(N_{33,n}^{(0)})^{-1} \right)^* = (N_{33,n}^{(0)})^{-1}N_{31,n}^{(0)}.$$

Moreover, the compatibility condition is precisely

$$\left(N_{24,n}^{(1)}(0)N_{44,n}^{(1)}(0)^{-1} \right)^* = N_{44,n}^{(1)}(0)^{-1}N_{42,n}^{(1)}(0).$$

Lemma 3.3 together with the fact that computing the adjoint is a continuous process in the weak operator topology ensures that M_1 and $M_{1'}$ satisfy the assumptions imposed on N_1 and $N_{1'}$ in Lemma 6.8. To estimate the norm bounds of M_1 and $M_{1'}$ in terms of $d, c, \sup\{\|N_n\|_\infty; n \in \mathbb{N}\}$ and ε , we use Lemma 6.1 and Lemma 6.10. Hence with the help of Lemma 6.8, we may estimate the constants of positive definiteness of $\mathcal{M}(0)$ and $\operatorname{Re} \mathcal{M}'(0)$ on $R(\mathcal{M}(0))$ and $N(\mathcal{M}(0))$, respectively, also in terms of $d, c, \sup\{\|N_n\|_\infty; n \in \mathbb{N}\}$ and ε . Note that we also have $R(\mathcal{M}(0)) = R(M(0))$. Remark 6.3 implies the remaining assertion. \square

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